TWO STEP RUNGE-KUTTA-NYSTRÖM METHOD FOR SOLVING SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

By

LATIFAH BINTI MD ARIFFIN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirement for the Degree of Doctor of Philosophy

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DEDICATIONS

To

My beloved parents, Mr. Md Ariffin Md Nor and Madam Mahfuzah Abd Ghaffar,

my faithful husband, Major Mohd Safiee Idris Mat Ali (Amdias),

my loyal and beautiful princesses,

Ms. Syasya Syahmina Amdias,

Ms. 'Adlina Safiyy Amdias,

Ms. 'Aaliah Syakirah Amdias and

future Amdias’s clan.
Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the Degree of Doctor of Philosophy

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Chairman: Professor Dato’ Mohamed Bin Suleiman, PhD
Faculty: Science

In this research, methods that will be able to solve the second order initial value problem (IVP) directly are developed. These methods are in the scheme of a multi-step method which is known as the two-step method. The two-step method has an advantage as it can estimate the solution with less function evaluations compared to the one-step method. The selection of step size is also important in obtaining more accurate and efficient results. Smaller step sizes will produce a more accurate result, but it lengthens the execution time.

Two-Step Runge-Kutta (TSRK) method were derived to solve first-order Ordinary Differential Equations (ODE). The order conditions of TSRK method were obtained by using Taylor series expansion. The explicit TSRK method was derived and its stability were investigated. It was then analyzed experimentally. The numerical results obtained were analyzed by making comparisons with the existing methods in terms of maximum global error, number of steps taken and function evaluations.

The explicit Two-Step Runge-Kutta-Nyström (TSRKN) method was derived with reference to the technique of deriving the TSRK method. The order conditions of TSRKN method were also obtained by using Taylor series expansion. The strategies in choosing the free parameters were also discussed. The stability of the methods derived were also investigated. The explicit TSRKN method was then analyzed experimentally and comparisons of the numerical results obtained were made with the existing methods in terms of maximum global error, number of steps taken and function evaluations.

Next, we discussed the derivation of an embedded pair of the TSRKN (ETSRKN) methods for solving second order ODE. Variable step size codes were developed and numerical results were compared with the existing methods in terms of maximum
global error, number of steps taken and function evaluations. The ETSRKN were then used to solve second-order Fuzzy Differential Equation (FDE). We observe that ETSRKN gives better accuracy at the end point of fuzzy interval compared to other existing methods.

In conclusion, the methods developed in this thesis are able to solve the system of second-order differential equation (DE) which consists of ODE and FDE directly.
Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Doktor Falsafah

KAEDAH RUNGE-KUTTA-NYSTRÖM DUA LANGKAH BAGI MENYELESAIKAN PERSAMAAN PEMBEZAAN BIASA PERINGKAT DUA

Oleh

LATIFAH BINTI MD ARIFFIN

Disember 2016

Pengerusi: Profesor Dato’ Mohamed Bin Suleiman, PhD
Fakulti: Sains


Seterusnya kami membincangkan penerbitan kaedah Benaman RKNDL (BRKNDL) bagi menyelesaikan PPB peringkat dua. Kod langkah berubah dibangunkan dan keputusan berangka dibandingkan dengan kaedah-kaedah sedia ada berdasarkan kepada ralat global maksimum, bilangan langkah dan penilaian fungsi. Kaedah BRKNDL ini
kemudiannya digunakan untuk menyelesaikan Persamaan Pembezaan Kabur (PPK). Kami mendapati bahawa kaedah BRKNDL memberi kejituan yang lebih baik pada titik hujung selang kabur berbanding dengan kaedah-kaedah sedia ada.

Kesimpulannya, kaedah-kaedah yang diterbitkan di dalam tesis ini dapat menyelesaikan sistem persamaan pembezaan (PP) yang merangkumi PPB dan PPK peringkat dua secara terus.
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</tr>
<tr>
<td>FCN</td>
<td>Number of Function Evaluations</td>
</tr>
<tr>
<td>FDE</td>
<td>Fuzzy Differential Equation</td>
</tr>
<tr>
<td>FETSRKN</td>
<td>Fuzzy Embedded Two-Step Runge-Kutta-Nyström</td>
</tr>
<tr>
<td>FIVP</td>
<td>Fuzzy Initial Value Problem</td>
</tr>
<tr>
<td>H-derivative</td>
<td>Hukuhara-Differentiability</td>
</tr>
<tr>
<td>HPC</td>
<td>High Performance Computing Machine</td>
</tr>
<tr>
<td>IVP</td>
<td>Initial Value Problem</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary Differential Equation</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
</tr>
<tr>
<td>PLTE</td>
<td>Principal Local Truncation Error</td>
</tr>
<tr>
<td>RK</td>
<td>Runge-Kutta</td>
</tr>
<tr>
<td>RK3(3)B</td>
<td>The three-stage third-order explicit RK method derived by Butcher (1987)</td>
</tr>
<tr>
<td>RK3(3)D</td>
<td>The three-stage third-order explicit RK method derived by Dormand (1996)</td>
</tr>
</tbody>
</table>
RK4(3)B The four-stage third-order embedded in four-stage fourth-order RK method derived by Butcher (1987)

RK4(3)F The four-stage third-order embedded in four-stage fourth-order RK method derived by Fehlberg (1970)

RKN Runge-Kutta-Nyström method

RKN3(3,12,3) A three-stage third-order RKN method with dispersive order twelve and dissipative order three derived by van der Houwen and Sommeijer (1987).

RKN4(4,10,5) A four-stage fourth-order RKN method with dispersive order ten and dissipative order five derived by van der Houwen and Sommeijer (1987)

TSRK Two-Step Runge-Kutta method

TSRK2(3) The two-stage third-order explicit TSRK method

TSRKN Two-Step Runge-Kutta-Nyström method

TSRKN2(3) The two-stage third-order explicit TSRKN method

TSRKN3(4) The three-stage fourth-order explicit TSRKN method

$C_2(p)$ Simplifying conditions for $m$-stage TSRKN method

$B_2(p)$ Simplifying conditions for $m$-stage TSRKN method

$B'_2(p)$ Simplifying conditions for $m$-stage TSRKN method
CHAPTER 1

INTRODUCTION

1.1 Introduction

Many problems in engineering and science can be formulated in terms of differential equations. These problems arise in mechanical and electrical systems, celestial and orbital mechanics, molecular dynamics, seismology and many other engineering problems. A differential equation is defined as an equation that involves a relation between an unknown function with one or more of its derivatives. Basically, a differential equation involving only ordinary derivatives with respect to single independent variable is called Ordinary Differential Equation (ODE). Meanwhile, a differential equation involving partial derivatives with respect to more than one independent variable is called Partial Differential Equation (PDE). Furthermore, ODE may be classified as either initial-value problem (IVP) or boundary-value problem (BVP).

The most discussed IVP are in class of the first and second order. These problems can be solved analytically when they are linear. However, very few nonlinear problems can be solved analytically. Thus, one must rely on numerical scheme to solve these problems. Methods for solving IVP numerically are classified into two schemes, which are the one-step method and the multi-step method. Many numerical one-step methods have been developed such as Euler method, Runge-Kutta (RK) method and Taylor series method where these methods are used to solve the first order IVP directly. These methods are also being used to solve the second order IVP indirectly by reducing it to the first order equation system. Even though this approach is easy to implement but it will enlarge the equation system and will increase the cost for the process.

1.2 Objectives of the Thesis

The main objective of this thesis is to develop a two-step Runge-Kutta-Nyström (TSRKN) method with a constant step-size and a variable step-size for solving special second-order IVP directly. The objectives can be accomplished by:

1. Develop the order conditions for two-step Runge-Kutta (TSRK) by using Taylor series expansion, derive the TSRK method and implement the method to solve first order IVP using constant step-size code;
2. Develop the order conditions for TSRKN by using Taylor series expansion, derive the TSRKN method and implement the method to solve special second order IVP using constant step-size code;
3. Investigate the stability and convergence of the derived TSRK and TSRKN methods;
4. Derive the embedded two-step Runge-Kutta-Nyström (ETSRKN) method and implement the method to solve special second order IVP using variable step-size code;

5. Solve second order fuzzy differential equations (FDE) by using ETSRKN method that had been derived previously to show the ability of the method to solve other type of DEs.

1.3 Outline of the Thesis

In Chapter 1, a brief introduction on differential equations and the application of numerical methods for solving different types of differential equations are given.

In Chapter 2, a brief introduction to IVP and Taylor series expansion were given. Then earlier researches related to TSRK and TSRKN methods for solving first order ODE and, second order ODE and FDE were provided. The stability properties for these methods were also presented. Some basic definitions and theorems related to the subject were also given. FDE and FIVP were discussed at the end of this chapter.

In Chapter 3, we start with the development of the order conditions from order one up to order four for TSRK method by using Taylor series expansion. Based on the order conditions obtained, we derived the two-stage third-order TSRK explicit method. The strategies of choosing the free parameters of the method for developing a more accurate computed solution are also discussed. The convergence of the method is proven and the stability regions of the method are presented. To illustrate the efficiency of the method, a number of tested problem are validated and the numerical results are compared with existing RK method of the same order derived by Dormand (1996) and Butcher (1987). Stability interval for all methods will also be presented.

Chapter 4 will discuss the development of order conditions from order one up to order four for TSRKN method by using Taylor series expansion. A two-stage third-order and three-stage fourth-order explicit TSRKN method were derived using the same strategy as found in Chapter 3. Several problems are solved and their numerical results are compared with the existing RK method of the same order. For existing RK method of order three, comparisons are made with methods derived by Butcher (1987) and van der Houwen and Sommeijer (1987). Likewise, comparisons are made with RK method of order four derived by Lambert (1991) and RKN method of order four derived by van der Houwen and Sommeijer (1987). Stability interval for all methods will also be presented.

For variable step-size, the development of an embedded pair for explicit TSRKN (ETSRKN) methods based on formulas derived in Chapter 4 are discussed in Chapter 5. The choices of free parameters in obtaining the optimized pair are also discussed. Special second-order IVP are solved including oscillating problems. Numerical results and their performances are presented. For the new ETSRKN 3(2) pair, comparisons are made with an existing embedded RK 3(2) pair derived by Dormand (1996).
Meanwhile, for the new ETSRKN 4(3) pair, comparisons are made with an existing embedded RK 4(3) pair derived by Butcher (1987) and Fehlberg (1970). The ETSRKN 4(3) pair method then is adapted for solving second-order fuzzy differential equations. Two fuzzy problems are solved and their numerical results are compared with the existing embedded RK method.

Finally, the summary of the whole thesis, conclusions and future research are given in Chapter 6.

1.4 Motivation and Contribution of the Thesis

Many differential equations which appear in practice are systems of second order IVP. This system can be reduced into first order differential equations of doubled dimension. In this study we are focusing on solving the second order IVP directly. Our proposed method able to solve the second order problems directly that is TSRKN method. We focus only on the explicit type of method. In addition to the implementation of the method, accuracy and stability are two other factors used for judging the efficacy of the methods.

1.5 Scope of the Thesis

This study concentrate on the development of new coefficient and efficient codes that are based on explicit TSRKN methods for numerical solution of IVP. These methods will then be used for solving system of second order ODEs directly for both constant and variable step size mode. The properties of this method will be analyzed in terms of order, consistence and convergence. Our main motivation is to reduce the number of steps taken in solving second order IVP directly by using this method as well as to reduce the number of function evaluations where it will ensure cost efficiency.
CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we begin with a brief introduction to IVP for second-order ODE in section 2.2. Next, Taylor series expansion were defined in section 2.3. The literature review for the TSREK method for solving first-order IVP is presented in section 2.4. Meanwhile, the literature review for the TSREKN method for solving special second-order IVP of ODE is presented in section 2.5. Section 2.6 defined the stability properties of both method. The TSREKN method were then proposed to solve the fuzzy differential equations with some modification in section 2.7 to solve Fuzzy Initial Value Problem (FIVP) in section 2.8.

2.2 Initial Value Problem

The initial value problem for a system of s special second order ODE is defined as

\[ y'' = f(x, y(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \] (2.1)

where \( y(x) = [y_1(x), y_2(x), ..., y_s(x)]^T \), \( y'(x) = [y'_1(x), y'_2(x), ..., y'_s(x)]^T \)

\[ f(x, y) = [f_1(x, y), f_2(x, y), ..., f_s(x, y)]^T, \quad x \in [a, b], \]

\( y_0 = [y_{01}, y_{02}, ..., y_{0s}]^T \) and \( y'_0 = [y'_{01}, y'_{02}, ..., y'_{0s}]^T \) are the vectors of initial conditions.

Theorem 2.1 (Existence and Uniqueness)
Let \( f(x, y) \) be defined and continuous for all points \((x, y)\) in the region \(D\) defined by \( a \leq x \leq b, -\infty < y < \infty, \ a \text{ and } b \text{ finite,} \) and let there exist a constant \( L \) such that, for every \( x, y, y' \) such that \((x, y)\) and \((x, y')\) are both in region \(D\), where

\[ |f(x, y) - f(x, y')| \leq L|y - y'|. \] (2.2)

Then, if \( y_0 \) is any given number, there exist a unique solution \( y(x) \) of the initial value problem (2.1), where \( y(x) \) is continuous and differentiable for all \((x, y)\) in \(D\).
The requirement (2.2) is known as Lipschitz condition, and the constant $L$ is known as a Lipschitz constant. For the proof of Theorem 2.1, see Henrici (1962). In this work, we shall assume the conditions of the theorem are satisfied, hence establishing the existence of a unique solution of (2.1).

2.3 Taylor Series Expansion

Given that the function $y(x)$ is sufficiently differentiable, $y(x + 2h)$ can be expended in a Taylor’s series form

$$y(x + 2h) = y(x) + 2hy'(x) + \frac{(2h)^2}{2!}y''(x) + \cdots + \frac{(2h)^p}{p!}y^p(x) + \cdots$$

(2.3)

where $y^p(x) = \frac{d^p y}{dx^p}$ with $p = 1, 2, \ldots$. Similarly, we write the Taylor’s series expansion of $y(x_n + 2h)$ as follows

$$y(x_n + 2h) = y(x_n) + 2hy'(x_n) + \frac{(2h)^2}{2!}y''(x_n) + \cdots + \frac{(2h)^p}{p!}y^p(x_n) + \cdots$$

(2.4)

The series on the right-hand side of (2.4) has an infinite number of terms in order to preserve the equality, and is not a practical formula for evaluating $y(x_{n+2})$. In practice, all terms up to and including that involving $h^p$ are included, that is

$$y(x_{n+2}) = y(x_n + 2h) = y(x_n) + 2hy''(x_n) + \frac{(2h)^2}{2!}y'''(x_n) + \cdots + \frac{(2h)^p}{p!}y^p(x_n) + 2h^{p+1}R_{p+1}(\xi_n)$$

(2.5)

where $R_{p+1}(\xi_n)$ is the remaining term with $x_n \leq \xi_n \leq x_n + 2h$, approximated by the following truncated series

$$y(x_{n+2}) = y(x_n + 2h) = y(x_n) + 2hy'(x_n) + \frac{(2h)^2}{2!}y''(x_n) + \cdots + \frac{(2h)^p}{p!}y^p(x_n) + O(h^{p+1}).$$

(2.6)

Equation (2.6) is the Taylor series method of order $p$, $y_n$ is taken to be the estimate of the exact value $y(x_n)$. From (2.6), generally an explicit two-step method can be written as

$$y_{n+2} = y_{n+1} + 2h\phi(x_n, y(x_n), h)$$

(2.7)
where $\emptyset(x, y, h)$ is a function of arguments $x, y, h$ and in addition, it depends on the right-hand side of (2.1). The function $\emptyset(x, y, h)$ is called the increment function. The true value $y(x_n)$ will satisfy
\[
y(x_{n+2}) = y(x_{n+1}) + 2h\emptyset(x_n, y(x_n), h) + T_n
\]
(2.8)
where $T_n$ is the truncation error.

**Definition 2.1** The method (2.7) is said to have order $p$ if $p$ is the largest integer that
\[
y(x + 2h) - y(x + h) - 2h\emptyset(x, y(x), h) = O(h^{p+1})
\]
(2.9)
where $y(x_n)$ is the analytical solution.

### 2.4 Two-Step Runge-Kutta (TSRK) Method

Consider the initial value problem for a system of ordinary differential equation (ODE)
\[
y'(x) = f(x, y(x)), \quad x \in [a, b], \quad y(x_0) = y_0,
\]
(2.10)
where the function $f: \mathbb{R}^q \to \mathbb{R}^q$ is assumed to be sufficiently smooth. These methods form a subclass of general linear methods considered by Butcher (1987) and are defined by
\[
y_{i+2} = (1 - \theta)y_{i+1} + \theta y_i + h \sum_{j=1}^{m} v_j f(x_i + c_j h, Y_i^j) + w_j f(x_{i+1} + c_j h, Y_{i+1}^j),
\]
\[
Y_i^j = y_i + h \sum_{s=1}^{m} a_{js} f(x_i + c_s h, Y_i^s), \quad j = 1, ..., m,
\]
\[
Y_{i+1}^j = y_{i+1} + h \sum_{s=1}^{m} a_{js} f(x_{i+1} + c_s h, Y_{i+1}^s), \quad j = 1, ..., m,
\]
(2.11)
i = 1, 2, ..., n - 1, \quad h = \frac{b-a}{n}, \quad \text{where the starting values } y_0 \text{ and } y_1 \text{ are assumed to be given (Jackiewicz and Renaut, 1995).}

It is convenient to represent (2.11) using the following Butcher table of coefficients:
Table 2.1: Butcher table for an explicit TSRK formula

<table>
<thead>
<tr>
<th>$c_i$</th>
<th>$a_{i1}$</th>
<th>$a_{i2}$</th>
<th>$\ldots$</th>
<th>$a_{im-1}$</th>
<th>$v_j$</th>
<th>$w_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$v_1$</td>
<td>$w_1$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$a_{31}$</td>
<td>$a_{32}$</td>
<td></td>
<td></td>
<td>$v_2$</td>
<td>$w_2$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$c_m$</td>
<td>$a_{m1}$</td>
<td>$a_{m2}$</td>
<td>$\ldots$</td>
<td>$a_{mm-1}$</td>
<td>$v_{m-1}$</td>
<td>$v_m$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$w_{m-1}$</td>
<td>$w_m$</td>
</tr>
</tbody>
</table>

where

$$c_i = \sum_{j=1}^{i-1} a_{ij}, \quad i = 1, 2, \ldots, m,$$

(2.12)

and

$$\sum_{j=1}^{m} (v_j + w_j) = 1 + \theta.$$

(2.13)

The TSRK methods were first introduced by Byrne and Lambert in 1966. These two step methods are different from the classical RK methods where the evaluations of $f$ in equation (2.10) made at the previous point were used along with those made at the current point in order to obtain the solution at the next point. They present a method having local accuracy $O(h^m)$ but requiring only $m-1$ derivative evaluations. They observed that these methods are consistent with the IVP and shown to be convergent to the exact solution of the IVP.

Jackiewicz et al. (1991) studied the implicit TSRK and derived the order conditions. Semi implicit TSRK was also constructed but no numerical results were presented for both types of method. In 1995, Jackiewicz et al. then analyzed the explicit formula for TSRK. They discovered that for order $p \leq 5$, the minimal number of stages for explicit TSRK method of order $p$ is equal to the minimal number of stages for explicit RK method of order $p-1$. For example, explicit TSRK method of order three only needs two stages compared to explicit RK method of order three which requires three stages. Meanwhile, Jackiewicz and Tracogna (1995) developed the general order conditions for a general class of TSRK. They derived the order conditions for TSRK by using
algebraic criteria introduced by Butcher (1987). The order conditions were generated using a mathematical program such as Maple™.

For stage \( m \leq 3 \), Williamson (1984) shows that the two-step method can achieve an extra degree of freedom compared to the one-step method. For instance, the three-stage two-step RK method can attain up to order four method while the one-step RK method at the same stage can attain up to order three method only. To obtain the order four methods, one-step RK will need at least four stages. Therefore we gain the extra degree of freedom without having extra function evaluations.

Byrne and Lambert (1966) had studied on the special case of method (2.11) when \( \theta = 0 \) and considered explicit TSRK with two-stage and three-stage methods of order three and four respectively. Meanwhile, van der Houwen and Sommeijer (1980), van der Houwen and Sommeijer (1982) and van der Houwen (1977) also studied explicit \( k \)-step \( m \)-stage Runge-Kutta (RK) methods. Furthermore, Renaut (1985) observed that method (2.11) was suitable for the numerical solution of system of ODE arising from the discontinuity of hyperbolic partial differential equation (PDE). Thus, Verwer (1976) applied the method of lines for the numerical integration of systems resulting from parabolic PDE.

### 2.5 Two-Step Runge-Kutta-Nyström (TSRKN) Method

TSRKN was derived as an indirect method from TSRK method (Paternoster, 2002). Paternoster showed the existence of P-stable methods within a class of TSRKN methods. Linear multistep methods have maximum order two in order to be P-stable (Coleman, 1992). The stability properties deteriorates when the order increases. Thus, derivation of a P-stable one stage second order method were presented. P-stability were defined as follows:

**Definition 2.2** The TSRKN method is P-stable if its interval of periodicity is \((0, +\infty)\).

(Paternoster, 2002)

In 2003, Paternoster describe the technique to derive TSRKN methods which integrate trigonometric and mixed polynomials. The methods depends on the parameter \( v = \omega h \) where \( \omega \) is the frequency and \( h \) is the step size. Paternoster then aim to analyze two step implicit method where its property has a high stage order which make them suitable for stiff systems. Paternoster still working on deriving the order conditions for TSRKN methods. A one-stage TSRKN method of order two were derived by Paternoster (2006). She lists the order conditions from order one to order two. But there is no numerical results presented in any of her publication. Thus, in this thesis the extension of TSRKN method for two stage and three stage are presented.

Consider the special second-order ODEs given by equation (2.1). The general \( m \)-stages form of TSRKN method introduced by Paternoster (2002) is defined by
\[ y_{i+2} = (1 - \theta)y_{i+1} + \theta y_i + h \sum_{j=1}^{m} v_j y'_i + h \sum_{j=1}^{m} w_j y'_{i+1} + h^2 \sum_{j=1}^{m} \bar{v}_j f(x_i + c_j h, Y'_i) + \bar{w}_j f(x_{i+1} + c_j h, Y'_{i+1}), \]

\[ y'_{i+2} = (1 - \theta)y'_{i+1} + \theta y'_i + h \sum_{j=1}^{m} v_j f(x_i + c_j h, Y'_i) + w_j f(x_{i+1} + c_j h, Y'_{i+1}), \]

where

\[ Y'_{i+1} = y_{i+1} + h c_j y'_{i+1} + h^2 \sum_{s=1}^{m} a_{js} f(x_{i+1} + c_s h, Y'_s, j = 1, \ldots, m, \]

\[ Y'_i = y_i + h c_j y'_i + h^2 \sum_{s=1}^{m} a_{js} f(x_i + c_s h, Y'_s, j = 1, \ldots, m. \]

(2.14)

\[ \theta, v_j, w_j, \bar{v}_j, \bar{w}_j, a_{js} \] for \( j, s = 1, \ldots, m \) are the coefficients of the methods.

Alternatively, according to Paternoster (2002) TSRKN (2.14) can be written as

\[ y_{i+2} = (1 - \theta)y_{i+1} + \theta y_i + h \sum_{j=1}^{m} v_j y'_i + h \sum_{j=1}^{m} w_j y'_{i+1} + h^2 \sum_{j=1}^{m} \bar{v}_j k_i^j + \bar{w}_j k_{i+1}^j, \]

\[ y'_{i+2} = (1 - \theta)y'_{i+1} + \theta y'_i + h \sum_{j=1}^{m} v_j k_i^j + w_j k_{i+1}^j, \]

where

\[ k_i^j = f \left( x_i + c_j h, y_i + h c_j y'_i + h^2 \sum_{s=1}^{m} a_{js} k_i^s \right), j = 1, \ldots, m, \]

\[ k_{i+1}^j = f \left( x_{i+1} + c_j h, y_{i+1} + h c_j y'_{i+1} + h^2 \sum_{s=1}^{m} a_{js} k_{i+1}^s \right), j = 1, \ldots, m. \]

(2.15)

TSRKN methods is said to need a lower number of stages for a given order of convergence in comparison with classical one-step RKN methods (Paternoster, 2003).
Equation (2.15) can be represented by the following Butcher array:

Table 2.2: Butcher table for an explicit TSRKN formula

| \( c_1 = 0 \) | \( a_{21} \) |
| \( c_2 \) | \( a_{31} \) | \( a_{32} \) |
| \( c_3 \) | \( \vdots \) | \( \vdots \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( c_m \) | \( a_{m1} \) | \( a_{m2} \) | \( \ldots \) | \( a_{m,m-1} \) |
| \( \theta \) | \( v_1 \) | \( v_2 \) | \( \ldots \) | \( v_{m-1} \) | \( v_m \) |
| \( \omega_1 \) | \( \omega_2 \) | \( \ldots \) | \( \omega_{m-1} \) | \( \omega_m \) |
| \( \bar{v}_1 \) | \( \bar{v}_2 \) | \( \ldots \) | \( \bar{v}_{m-1} \) | \( \bar{v}_m \) |
| \( \bar{\omega}_1 \) | \( \bar{\omega}_2 \) | \( \ldots \) | \( \bar{\omega}_{m-1} \) | \( \bar{\omega}_m \) |

The following definitions are given for method 2.14 by Paternoster (2003):

**Definition 2.3** An \( m \)-stage TSRKN method is said to satisfy the simplifying conditions \( C_2(p) \) if its parameters satisfy

\[
\sum_{s=1}^{m} a_{js} c_s^{k-2} = \frac{c_j^k}{k(k-1)}, \quad j = 1, \ldots, m, \quad k = 1, \ldots, p.
\]

**(2.16)**

**Definition 2.4** An \( m \)-stage TSRKN method is said to satisfy the simplifying conditions \( B_2(p) \) if its parameters satisfy

\[
\sum_{j=1}^{m} \left( v_j (c_j - 1)^{k-2} + w_j c_j^{k-2} \right) = \frac{1 - (-1)^k \theta}{k(k-1)} - \frac{1}{k-1} \sum_{j=1}^{m} v_j',
\]

\[
\quad j = 1, \ldots, m, \quad k = 1, \ldots, p.
\]

**(2.17)**

**Definition 2.5** An \( m \)-stage TSRKN method is said to satisfy the simplifying conditions \( B'_2(p) \) if its parameters satisfy
\[
\sum_{j=1}^{m} \left( v_j'(c_j - 1)^{k-2} + w_j'c_j^{k-2} \right) = \frac{1 - (-1)^k \theta}{k(k - 1)}, \quad k = 1, \ldots, p.
\]

(2.18)

The simplifying assumptions \( C_z(p), B_z(p) \) and \( B'_z(p) \) given in Definition 2.3-2.5 allow the reduction of order conditions for TSRKN methods. However, these conditions are sufficient conditions for TSRKN method to have order \( p \), but not necessarily (Paternoster, 2003).

### 2.6 Stability Properties of TSRK and TSRKN methods

TSRK and TSRKN method possess their own stability polynomial. Derivation of TSRK stability polynomial will be presented in Chapter 3 and derivation of TSRKN stability polynomial will be presented in Chapter 4. However, both method have the same properties as given by the following definitions and theorem:

**Definition 2.6** The method (2.11) and (2.14) is said to satisfy the root condition if all the roots of characteristic polynomial have modulus less than or equal to unity (within or on the unit circle), and those of modulus unity are simple.

**Definition 2.7** The method (2.11) and (2.14) is said to be zero-stable if it satisfies the root condition.

**Theorem 2.2** The necessary conditions for the method (2.11) and (2.14) to be convergent are that it must be both consistent and zero-stable. The method is consistent if it has at least order 1.

(Watt, 1967)

**Definition 2.8** The method is said to be absolutely stable for a given roots if all the roots lies within the unit circle.

**Definition 2.9** The method is zero stable if it satisfied \(-1 < \theta \leq 1\).

(Jackiewicz et al., 1991)

### 2.7 Fuzzy Differential Equation

Fuzzy Differential Equation (FDE) or Fuzzy Initial Value Problem (FIVP) are used for modeling problems in science and engineering. Most of the problems require the solution of FDE which satisfies fuzzy initial conditions. Initially, the derivative of fuzzy-valued functions was first introduced by Chang and Zadeh (1972). It was followed by Dubois and Prade (1982) who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu (1983) where they had generalized and extended the concept of Hukuhara-differentiability (H-derivative)

Buckley et al. (2002), Kaleva (1990) and Ma et al. (1999) considered the one-step method for FIVP. Meanwhile, Dahaghin and Moghadam (2010) had used a modified two-step Simpson method for solving fuzzy differential equations. The convergence theorem and the numerical results also shows that for a smaller step size $h$, smaller errors are obtained and hence yields a better result.

In this section we give some basic definitions and properties of fuzzy sets obtained by Zadeh (1965), Friedman et al. (1998), Kanagarajan and Sambath (2010), Rabiei (2012), Friedman et al. (1999), Sedagatfar et al. (2013), Bede and Gal (2005), Chalco-cano and Roman-Flores (2006), and James and Thomas (2001).

**Definition 2.10** A fuzzy set $A$ in $X$ where $X$ is a space of points with a generic element, $x$ is characterized by characteristic function $f_A(x)$ which associates with each point in $X$ where $X$ is a real number in the interval $[0,1]$.

(Zadeh, 1965)

**Definition 2.11** An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$ for all $r \in [0,1]$, which satisfy the following requirements:

- $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$;
- $\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0,1]$;
- $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

(Friedman et al., 1998)

**Definition 2.12** A fuzzy number, $\mu$ is called a triangular fuzzy number if defined by three numbers $\alpha < \beta < \gamma$ where the graph of $\mu(x)$ is a triangle with the base on the interval $[\alpha, \gamma]$ and vertex at $x = \beta$ and its membership function has the following form:

$$
\mu(x; \alpha, \beta, \gamma) = \begin{cases} 
0, & \text{if } x < \alpha, \\
\frac{x - \alpha}{\beta - \alpha}, & \text{if } \alpha \leq x \leq \beta, \\
\frac{\gamma - x}{\gamma - \beta}, & \text{if } \alpha \leq x \leq \beta, \\
0, & \text{if } x > \gamma,
\end{cases}
$$

and its $r$-level is

$$
[\mu]_r = [\alpha + r(\beta - \alpha), \gamma - r(\gamma - \beta)], r \in [0,1].
$$

(Kanagarajan and Sambath, 2010)
Definition 2.13 Let $A$ be a fuzzy interval defined in $R$. The $\alpha$ cut of $A$ is the crisp set $[A]^\alpha$ that contains all elements in $R$ such that the membership values of $A$ is greater than or equal to $\alpha$, that is

$$[A]^\alpha = [a_1^\alpha, a_2^\alpha], \alpha \in (0,1].$$

(Rabiei, 2012)

Definition 2.14 Let $f: R \to E$ be a fuzzy valued function. If for arbitrary fixed $t_0 \in R$ and $\varepsilon > 0, \delta > 0$ such that

$$|t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \varepsilon,$$

$f$ is said to be continuous.

(Friedman et al., 1999)

Definition 2.15 Let $x, y \in E$. If there exist $z \in E$ such that $x = y + z$, then $z$ is called the $H$-difference of $x$ and $y$ and it is denoted by $x \ominus y$.

(Sedagatfar et al., 2013)

Definition 2.16 Let $f: (a, b) \to E$ and $x_0 \in (a, b)$. We say that $f$ is strongly generalized $H$-differentiable at $x_0$. If there exists an element $f'(x_0) \in E$, such that:

1) for all $h > 0$ sufficiently near to 0, $\exists f(x_0 + h) \ominus f(x_0), \exists f(x_0) \ominus f(x_0 - h)$ such that the following limits hold

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

2) for all $h < 0$ sufficiently near to 0, $\exists f(x_0) \ominus f(x_0 + h), \exists f(x_0 - h) \ominus f(x_0)$ such that the following limits hold

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus f(x_0)}{h} = f'(x_0).$$

(Bede and Gal, 2005; Chalco-cano and Roman-Flores, 2006)

In the special case when $f$ is a fuzzy-valued function, we have the following results.

Theorem 2.3 Let $f: R \to E$ be a function and denote $f(t) = \left(f(t;r), \bar{f}(t;r)\right)$, for each $r \in [0,1]$. Then

1) if $f$ is differentiable in the first form 1) in Definition 2.13 then $f(t;r)$ and $\bar{f}(t;r)$ are differentiable functions and $f'(t) = \left(f'(t;r), \bar{f}'(t;r)\right)$,

2) if $f$ is differentiable in the first form 2) in Definition 2.13 then $f(t;r)$ and $\bar{f}(t;r)$ are differentiable functions and $f'(t) = \left(\bar{f}'(t;r), f'(t;r)\right)$.

(James and Thomas, 2001)

2.8 Fuzzy Initial Value Problem

Consider the second-order FIVP as follows:
\[ y''(t) = f(t, y(t)), \quad y(t_0) = y_0, \]
\[ y'(t_0) = y'_0, \quad t \in [t_0 T], \]

where \( f \) is a fuzzy function with \( r \)-level sets of initial values

\[ [y_0]_r = [y_1(0, (t; r)), y_2(0, (t; r))], \]
\[ [y_0']_r = [y'_1(0, (t; r)), y'_2(0, (t; r))], \quad r \in [0, 1]. \]

We write

\[ y(t, y) = [y_1(t; r), y_2(t; r)], \quad y'(t, y) = [y'_1(t; r), y'_2(t; r)], \]
\[ f(t, y(t; r)) = [f_1(t, y(t; r)), f_2(t, y(t; r))], \]

where:

\[ f_1(t, y(t; r)) = F[t, y_1(t; r), y_2(t; r)], \]
\[ f_2(t, y(t; r)) = G[t, y_1(t; r), y_2(t; r)]. \]

By using the extension principle, when \( y(t) \) is a fuzzy number we have the membership function

\[ f(t, y(t))(s) = \sup\{y(t)(\tau) | s = f(t, \tau)\}, s \in \mathbb{R}. \]

It follows that

\[ [f(t, y(t))]_r = [f_1(t, y(t; r)), f_2(t, y(t; r))], \quad r \in [0, 1], \]

where

\[ f_1(t, y(t; r)) = \min\{f(t, u) | u \in [y_1(t; r), y_2(t; r)]\}, \]
\[ f_2(t, y(t; r)) = \max\{f(t, u) | u \in [y_1(t; r), y_2(t; r)]\}. \]
CHAPTER 3

SOLVING FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY EXPLICIT TWO-STEP RUNGE-KUTTA (TSRK) METHOD USING CONSTANT STEP SIZE

3.1 Introduction

Consider the initial value problem for a system of ordinary differential equations (ODEs) given previously as equation (2.10) in Chapter 2. We will investigate the explicit two-step Runge-Kutta (TSRK) method for the numerical solution of (2.10). These methods form a subclass of general linear methods considered by Butcher (1987) and are given as equation (2.11) in Chapter 2. In this chapter we will derive the two-stage TSRK method with algebraic order three.

From method (2.11), by making the interpretation

$$k_i^j = f(x_i + c_j h, y_i^j)$$

(3.1)

method (2.11) can be re-written as the following:

$$y_{i+2} = (1 - \theta)y_{i+1} + \theta y_i + h \sum_{j=1}^{m} v_j k_i^j + w_j k_{i+1}^j,$$

$$k_i^j = f \left( x_i + c_j h, y_i + h \sum_{s=1}^{m} a_{js} k_i^s \right), \quad j = 1, ..., m,$$

$$k_{i+1}^j = f \left( x_{i+1} + c_j h, y_{i+1} + h \sum_{s=1}^{m} a_{js} k_{i+1}^s \right), \quad j = 1, ..., m.$$  

(3.2)

We considered the minimizations of the norms of the Principal Local Truncation Error (PLTE) coefficients because they lead to an accurate method. Minimization of the truncation error coefficients are defined in Dormand (1996) and given by

$$\|\tau^{(p+1)}\|_2 = \sqrt{\sum_{j=1}^{n_{p+1}} \left( \tau_j^{(p+1)} \right)^2}.$$  

(3.3)
In this thesis the TSRK\(m(p)\) notation will be used where

- \(m\) – the stage of the method, and
- \(p\) – the algebraic order of the method.

### 3.2 Derivation of Order Conditions

To find the order conditions of the method, Taylor’s series expansion were used. From method (3.2) TSRK with \(\theta = 0\) is given by

\[
y_{i+2} = y_{i+1} + h \sum_{j=1}^{m} v_j k_i^j + w_j k_{i+1}^j,
\]

\[
k_i^j = f \left( x_i + c_j h, y_i + h \sum_{s=1}^{m} a_{js} k_i^s \right), \quad j = 1 \ldots m,
\]

\[
k_{i+1}^j = f \left( x_{i+1} + c_j h, y_{i+1} + h \sum_{s=1}^{m} a_{js} k_{i+1}^s \right), \quad j = 1 \ldots m.
\]

(3.4)

Let

\[
x_{i+1} = x_i + h \quad \text{and} \quad y_{i+1} = y_i + h y'_i.
\]

(3.5)

Substitute (3.5) into (3.4), we have

\[
y_{i+2} = y_{i+1} + h y'_i + h \sum_{j=1}^{m} v_j k_i^j + w_j k_{i+1}^j,
\]

\[
k_i^j = f \left( x_i + c_j h, y_i + h \sum_{s=1}^{m} a_{js} k_i^s \right), \quad j = 1 \ldots m,
\]

\[
k_{i+1}^j = f \left( x_i + (1 + c_j) h, y_{i+1} + h \sum_{s=1}^{m} a_{js} k_{i+1}^s \right), \quad j = 1 \ldots m,
\]

(3.6)

with \(0 \leq c_j \leq 1\). We apply the row sum condition of RK method to (3.6) where

\[
c_i = \sum_{j=1}^{i-1} a_{ij}, \quad i = 1 \ldots m.
\]

(3.7)
With reference to Williamson (1984), we applied the Taylor’s theorem to method (3.6) to obtain the order conditions by the following steps:

**Step 1** Expand the term
\[ \sum_{s=1}^{m} a_{js}k_i^j \]
for \( k_i^j \) from equation (3.6) with the respective value of \( m \) (number of stage).

**Step 2** Expand \( k_i^j \) by using Taylor’s expansion series
\[
f(x_0 + h, y_0 + k) = f(x_0, y_0) + hf_x(x_0, y_0) + k f_y(x_0, y_0) + \frac{1}{2!} \left( h^2 f_{xx}(x_0, y_0) + 2hk f_{xy}(x_0, y_0) + k^2 f_{yy}(x_0, y_0) \right) + \cdots
\]

**Step 3** Assume \( k_i^j = A_j + hB_j + h^2C_j + \cdots + O(h^{m+1}) \) then substitute into \( k_i^j \) in Step 2.

**Step 4** Equate powers of \( h \) by letting \( k_i^j = f(x_0 + h, y_0 + k) \). We will obtain the terms for \( A, B, C, \ldots \) respectively.

**Step 5** Write the expansion for \( y(x_i + 2h) \) with the following form
\[
y(x_i + 2h) = y(x_i) + 2hy'(x_i) + 2h^2y''(x_i) + \frac{4}{3} h^3y'''(x_i) + \cdots
\]
where some of the differential term were given by
\[
y'(x_i) = f(x, y) = f
y''(x_i) = f_x + ff_y = F
y'''(x_i) = f_{xx} + 2ff_{xy} + f^2f_{yy} + f_y(f_x + ff_y) = G + f_y F
\]

**Step 6** Repeat Step 1-Step 5 for \( k_{i+1}^j \).

**Step 7** Substitute \( k_{i+1}^j \) and \( k_i^j \) into \( y_{i+2} \) and \( y_{i+2}' \).

**Step 7** Construct the general form of Taylor series for both \( y_{i+2} \) and \( y_{i+2}' \).

**Step 8** Compare \( y_{i+2} \) and \( y_{i+2}' \) from Step 7 with Step 5.

Expanding the Taylor’s series for several variable functions will lead to difficulties to solve tedious and multiple long equations. Referring to works done by Gander and Gruntz (1999), the Taylor’s series expansion can be done by using computer algebra systems such as Maple\textsuperscript{TM}, Mathematica, MuPad, or any other computer algebra system. With the above algorithm, we applied the proposed expansion technique by using Maple\textsuperscript{TM} to derive the order conditions for method (3.6). Thus, we obtained the following order conditions for \( y_{i+2} \) up to order four:
Order 1:
\[
\sum_{i=1}^{m} v_i + w_i = 1
\]  
(3.8)

Order 2:
\[
\sum_{i=1}^{m} w_i + c_i(v_i + w_i) = \frac{3}{2}
\]  
(3.9)

Order 3:
\[
\frac{1}{2} \sum_{j=1}^{i-1} w_i (2a_{ij} + 1) = \frac{7}{6} (i = 1 \ldots m)
\]  
(3.10)

\[
\frac{1}{2} \sum_{i=1}^{m} 2w_ic_i + c_i^2(v_i + w_i) + w_i = \frac{7}{6}
\]  
(3.11)

Order 4:
\[
\frac{1}{6} \sum_{j=1}^{i-1} w_i (1 + 3a_{ij}) = \frac{5}{8} (i = 1 \ldots m)
\]  
(3.12)

\[
\frac{1}{6} \sum_{i=1}^{m} w_i (1 + 2c_i + 2c_i^2) + c_i^3(v_i + w_i) = \frac{5}{8}
\]  
(3.13)

\[
\frac{1}{2} \sum_{i=1}^{m} w_i (1 + c_i + 2a_{ij} + 2c_i a_{ij}) = \frac{15}{8}
\]  
(3.14)
3.3 Derivation of Two-Stage Third-Order Explicit TSRK Method

We have four equations ((3.8)-(3.11)) with six unknowns to be solved. We are left with two degrees of freedom. Therefore, the following solution of two-parameter family is obtained with $a_{21}$ as a free parameter.

\[
c_2 = a_{21}, \quad v_1 = -\frac{1}{3} \left(\frac{3a_{21} + 2}{a_{21}}\right), \quad v_2 = -\frac{2}{3} \left(\frac{1 + 2a_{21}}{a_{21}(a_{21} - 1)}\right),
\]

\[
w_1 = \frac{2}{3} \left(\frac{-2a_{21} + 3a_{21}^2 + 2}{a_{21}(a_{21} - 1)}\right), \quad w_2 = \frac{4}{3a_{21}}.
\]

Since we are deriving the method of order three, we need the order conditions for order four so as to minimize the PLTE as proposed by Dormand (1996) to achieve a particular order of accuracy. By using the \texttt{minimize} command in Maple\textsuperscript{TM}, $\|\tau^{(4)}\|_2$ has a minimum value at $a_{21} = 0.38095238095238095237$. The obtained value of the free parameters gives $\|\tau^{(4)}\|_2 = 0.93151750800769586224$. The coefficients in Table 3.2 are generated using Maple\textsuperscript{TM} where the significant digits is set to 20 by command Digits.

**Table 3.1: Coefficients for TSRK2(3) method**

<table>
<thead>
<tr>
<th>$c_2$</th>
<th>0.38095238095238095237</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>-2.750000000000000000000000</td>
</tr>
<tr>
<td>-4.7307692307692307692</td>
<td>3.5000000000000000000000001</td>
</tr>
</tbody>
</table>

3.4 Derivation of Stability Polynomial for TSRK Method

Dekker and Verwer (1984) propose to apply the test equation $y' = \lambda y$ and by replacing $f(x, y) = \lambda y$ into method (3.6) we obtain

\[
y_i + 2 = y_i + h(\lambda y_i) + h \sum_{j=1}^{m} v_j k_i^j + w_j k_{i+1}^j,
\]

(3.15)

\[
k_i^j = \lambda \left(y_i + h \sum_{s=1}^{m} a_{js} k_i^s\right), \quad j = 1, \ldots, m,
\]

(3.16)
\[ k_{i+1}^j = \lambda \left( y_i + h(\lambda y_i) + h \sum_{s=1}^{m} a_{js}k_{i+1}^s \right), \quad j = 1, \ldots, m. \]  

(3.17)

The application of the test equation to equation (3.16) yields the recursion

\[ k_i = \lambda y_i u + \lambda h Ak_i \quad (i = 1, \ldots, m), \]

(3.18)

where by

\[ u = [1, \ldots, 1]^T, A = [a_{ij}], k_i = [k_1, k_2, \ldots, k_m]. \]

From equation (3.18), \( k_i \) can also be written as

\[ k_i = (I - HA)^{-1} \lambda y_i u, \]

(3.19)

where by

\[ H = \lambda h. \]

(3.20)

Similar process for equation (3.17), we obtained

\[ k_{i+1} = y_i (\lambda (I - HA)^{-1} u)(I + H). \]

(3.21)

Substitute equation (3.19), (3.20) and (3.21) into equation (3.15) yields

\[ y_{i+2} = (I + H)y_i + Hw^T(I - HA)^{-1}y_i u + Hw^T(I - HA)^{-1}y_i u + H^2 w^T(I - HA)^{-1}y_i u. \]

(3.22)

or of the form

\[ y_{i+2} = R(H)y_i, \]

(3.23)

where by
\[
\frac{y_{i+2}}{y_i} = R(H) = (I + H) + Hv^T(I - HA)^{-1}u + Hw^T(I - HA)^{-1}u
\]
\[
+ H^2w^T(I - HA)^{-1}u.
\]
(3.24)

The stability function associated with this method is given by
\[
\phi(\psi, H) = \psi^2I - R(H).
\]
(3.25)

Substitute all coefficient values obtained in Table 3.1 into equation (3.24) yields the following stability polynomial for TSRK2(3) method;
\[
R(H) = 1 + 2.000000000001H + 2.0000000001H^2 + 1.333333334H^3,
\]
and its characteristic equation can be written as
\[
\phi(\psi, H) = \psi^2 - 1 - 2.000000000H - 2.0000000001H^2 - 1.333333334H^3.
\]

![Figure 3.1: Stability region for TSRK2(3) method](image)

Figure 3.1 shows the stability plot of the third order formula used to compute the solution at \(x_{n+2}\). The absolute stability interval obtained is approximately \((-1.256372, 0.0)\) which lies within the close region in Figure 3.1. TSRK2(3) method is zero stable according to Definition 2.9 given in Chapter 2 since it was derived with \(\theta = 0\). Furthermore, all roots of stability polynomial for TSRK2(3) are less than or equals to one within the stability interval. Since this method is of order three, then it is consistent and leads to a convergent method (Watt, 1967).
Table 3.2: Stability interval for TSRK2(3), RK3(3)D and RK3(3)B method

<table>
<thead>
<tr>
<th>Method</th>
<th>Stability Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>TSRK2(3)</td>
<td>(−1.256372, 0.0)</td>
</tr>
<tr>
<td>RK3(3)D</td>
<td>(−2.511999, 0.0)</td>
</tr>
<tr>
<td>RK3(3)B</td>
<td>(−2.512699, 0.0)</td>
</tr>
</tbody>
</table>

Table 3.2 summarized the stability interval for TSRK2(3) method and three-stage third order RK methods derived by Dormand (1996) (RK3(3)D) and Butcher (1987) (RK3(3)B). Meanwhile Figure 3.2-3.3 shows the stability region for RK3(3)D and RK3(3)B method. From Table 3.2 and Figure 3.2-3.3, observed that the stability interval/region for both RK3(3)D and RK3(3)B method are larger compared to TSRK2(3) method. The difference between these two existing methods are that the have different coefficients values (refer to Dormand, 1996 and Butcher, 1987 for the full literature).
3.5 Problems Tested

In this section validation of the derived method solving various types of initial value problem (IVP) were given.

**Problem 3.1**

\[ \frac{dy}{dx} = -y(x), \quad y(0) = 1, \quad 0 \leq x \leq 20. \]

Exact solution:

\[ y(x) = e^{-x}. \]

Source: Davis (1963)

**Problem 3.2 (A special case of the Riccati equation)**

\[ \frac{dy}{dx} = \frac{-y^2(x)}{2}, \quad y(0) = 1, \quad 0 \leq x \leq 20. \]

Exact solution:

\[ y(x) = \frac{1}{\sqrt{x + 1}}. \]

Source: Davis (1963)
Problem 3.3 (A logistic curve)

\[ \frac{dy}{dx} = \frac{y(x)}{4} \left( 1 - \frac{y(x)}{20} \right), \quad y(0) = 1, \quad 0 \leq x \leq 20. \]

Exact solution:

\[ y(x) = \frac{20}{1 + 19e^{-x/4}} \]

Source: Davis (1963)

Problem 3.4

\[ \frac{dy}{dx} = y(x) \cos(x), \quad y(0) = 1, \quad 0 \leq x \leq 20. \]

Exact solution:

\[ y(x) = e^{\sin(x)} \]

Source: Hull et al. (1972)

3.6 Numerical Results

The numerical results of our third-order method are tabulated in Tables 3.3 to 3.6. One measure of the accuracy of a method with constant step size is to examine the maximum error, \( \text{MaxE}(T) \) which is defined by

\[ \text{MaxE}(T) = \max \| y(x_n) - y_n \|, \quad x_n = x_0 + 2h. \]

Tables 3.3 to 3.6 shows the absolute maximum error for our third-order method when solving Problem 3.1 to 3.4 with five different step values,

\[ h = \frac{1}{10^i}, \quad i = 2, \ldots, 5. \]

(Notation: \( 1.234567(-4) \) means \( 1.234567 \times 10^{-4} \).) The codes for this method were run using Code Block where the time measured are up to three decimal points. In addition, the following abbreviations will be used in Tables 3.3 to 3.6.

- TSRK2(3): The two-stage third-order explicit TSRK method.
- \( h \): Step size.
- STEPS: Number of steps taken.
- FCN: Number of function evaluations.
- MAX E: Maximum error.
- TIME: Time taken to solve in seconds, s.


