AN ANALYTICAL STUDY ON THE RESIDUAL STRENGTH OF RC STRUCTURES WITH DEGRADATION DAMAGE

Hamidun bin Mohd Noh

GRADUATE SCHOOL OF ENGINEERING
KYUSHU UNIVERSITY
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Hamidun Bin Mohd Noh

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Hamidun
Hamidun Bin Mohd Noh
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ABSTRACT

Nowadays, rebar corrosion has become one of the biggest threats in reinforced concrete (RC) industry due to chemical and mechanical attacks. The main consequences of this phenomenon include a loss of cross section of steel area and the induced of expansive pressure due to volume expansion which caused cracking of concrete, spalling and delaminating of the concrete cover. Thus, it reduces the bond strength between steel reinforcing bar and concrete, and deteriorates the strength of the whole structure. A continuous process of rebar corrosion is not just shorten the service life, reduce the safety and serviceability level, but can also increase the maintenance cost. For the purpose of maintaining safety and serviceability, it is necessary to evaluate the durability of existing structures accurately, in order to predict the structure’s rate of deterioration and its future strength. In this study, an experimental work of electrolytic corrosion process was conducted for several levels of rebar corrosion. Next, a static loading test was adopted to assess the structural performance and obtain the residual strength of the beams. Meanwhile, continuum damage mechanics were utilized in analysis of damage caused by chemical and mechanical effects. Within the framework of this method, chemical damage caused by rebar corrosion was considered. Then the effects of chemical and mechanical damage were calculated by introducing two independent scalar damage variables into the constitutive equation. In order to calculate the chemical damage evolution, the diffusion process of chloride ions that impact the rebar corrosion in concrete was simulated, and an evaluation was conducted on an affected cross-sectional area of a steel bar. The proposed method was found enable to validate the experiment’s results and predict the future strength of RC structural members under various exposition periods. In addition, the comparison carried out between the isotropic and orthotropic conditions confirmed the importance of orthotropic analysis in order to obtain the worst-case scenario of the structure. Moreover, the dead load and the hydrostatic stress effects were also investigated and the predominant factor of dead load in the length of a structure span was determined and it was found that the dead load of a structure is dominant in increasing a structure’s span length.
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CHAPTER I

INTRODUCTION

1.1 An overview on deterioration of existing RC structure due to corrosion

Nowadays, corrosion of reinforcing bar has become one of the biggest for reinforced concrete (RC) structure. Rebar corrosion appears on the surface of steel reinforcements once the accumulation of chloride ions exceeds the threshold level. At the early stage, rebar corrosion reduces the effective cross-sectional area of the steel reinforcement. Then, the accumulation of rebar corrosion product causes volume expansion and exerts expansive pressure to the concrete surface. Moreover, the rebar corrosion weakens the bond between steel and concrete, which decreases the stiffness and increases the level of deterioration of a structure. A continuous process of rebar corrosion is not just shorten the service life, reduce the safety and serviceability level, but also increase the maintenance cost.

There are several factors affecting the durability of concrete structures. The durability of concrete structures is influenced by mix proportion factors such as cement type, casting method, curing process and other execution conditions (Tsutsumi et al., 1997). Besides, environmental factors such as the proximity of the structure to the sea, climate and temperature, humidity and other geomorphic conditions contribute much to the invasion of chloride ions on concrete
surfaces (Al-Rabiah et al., 1990). Moreover, geometrical factors such as the ratio of concrete cover to bar diameter has a significant effect on durability (Hamidun and Sonoda, 2015(a); Du et al., 2006; Williamson and Clark, 2000). Pengwei et al. (2011) conducted a finite element analysis, using a formula predicting the cracking of concrete covers taking into account the cover depth, rebar diameter, tensile strength and elastic modulus of the concrete.

The properties of oxide layers as corrosion products also have a great influence on corrosion cracking in concrete. In this aspect, the most influential factor is the oxide expansion ratio, which depends on the specific type of oxide formed. Depending on the level of oxidation, the volume increase due to rebar corrosion is commonly around 2.0. However, up to 6.5 times (Table 1) of the original iron volume is consumed by the rebar corrosion process due to the formation of various corrosion products (Mehta and Monteiro, 1997). Nevertheless, Molina et al. (1993) claimed that the effective expansion ratio maybe less than that corresponding to a given type if the oxide diffuse in the porous structure of the concrete.

Table 1: Correlation between ‘α’ and ‘α1’ for various corrosion products

(Mehta and Monteiro, 1997)

<table>
<thead>
<tr>
<th>Name of corrosion products</th>
<th>FeO</th>
<th>Fe$_3$O$_4$</th>
<th>Fe$_2$O$_3$</th>
<th>Fe(OH)$_2$</th>
<th>Fe(OH)$_3$</th>
<th>Fe(OH)$_3$.3H$_2$O</th>
</tr>
</thead>
<tbody>
<tr>
<td>α’</td>
<td>0.777</td>
<td>0.724</td>
<td>0.699</td>
<td>0.622</td>
<td>0.523</td>
<td>0.347</td>
</tr>
<tr>
<td>α1’</td>
<td>1.80</td>
<td>2.00</td>
<td>2.20</td>
<td>3.75</td>
<td>4.20</td>
<td>6.40</td>
</tr>
</tbody>
</table>

where: α, ratio of molecular weight of iron to the molecular weight of corrosion products; α1, ratio of volume of corrosion products to the volume of iron consumed in the corrosion process.

On the other hand, the dead load and the hydrostatic stress effect also have a noticeable influence on the deterioration of RC structures. A study conducted by Wang (2011) found that dead loads (static in location and magnitude) reduce the porosity of concrete in compressive regions, contributing to the reduction of the diffusion coefficient as well as impeding the invasion of chloride ions. Meanwhile, the porosity of concrete in tensile areas encourages chloride ions penetration due to the incrementation of the diffusion coefficient.
Reinforced concrete bars become corroded either due to diffusion and ingress of chloride ions or due to carbonation process. However, the effects of chloride ions ingress are marked and more predominant than those of carbonation (Almusallam, 2001; Francois et al., 2013). A numerous studies on chloride ions-induced rebar corrosion were already conducted. Many researchers considered a two phases model of service life proposed by Tutti (1982) to predict the deterioration of RC structure. The first phase is the initiation stage which considered the time required for chloride ions to diffuse and penetrate to the steel surface and initiate rebar corrosion. While the second phase considered the time between initiations of rebar corrosion and cracking of concrete. Besides, the improved and extended models to predict the deterioration process are widely explored. Bastidas-Arteaga et al. (2009) suggested the corrosion-fatigue deterioration process into three stages which are rebar corrosion initiation and pit nucleation, pit-to-crack transition and crack growth stage. Meanwhile, Pan and Wang (2011) proposed a three phases model which consist of chloride ions ingress, rebar corrosion and concrete cracking. On the other hand, a micromechanics approach to predict the service life in four stages: initiation stage, propagation stage, acceleration stage and deterioration stage was introduced by Song et al. (2007).

Apart of chloride ions ingress process, the effect of external load to the structure also can initiate rebar corrosion. The mechanical loading or external load impact on the structure can cause cracking. Due to the generation of cracks, the chloride ions can easily penetrate into the concrete and accumulate on reinforcement surface. The accumulation of chloride ions will destroy the high alkaline environment and disrupt the passivated film on the steel surface. The rebar corrosion is initiated when sufficient oxygen and moisture are present.
1.2 Problem definition

The deterioration of RC structures due to rebar corrosion is a serious problem among construction stakeholders. As early as 1979, Bazant conducted a study on the effect of steel rebar corrosion in concrete structures. The effect covers not only the reduction of service life, but also the degradation of safety and serviceability. Moreover, maintenance costs have been increasing over the years. Due to the serious effects of rebar corrosion on RC structures, numerous studies have been carried out until now, seeking a better solution.

Investigations of durability, followed by predicting the future strength of concrete structures, are important issues to highlight in order to improve efficiency in safety, serviceability and performance. In order to evaluate the lifetime accurately, degradation process due to mechanical and chemical phenomenon need to be considered at the same time.

1.3 Research objectives

The aim of this study is to evaluate the durability of reinforced concrete structure under the damage caused by the chloride ions attacks and to predict the deterioration level of structure and their future strength.

The specific objectives of this study are as follows:

1. To implement a proper constitutive equation and develop an analysis procedure by using mechanical and chemical damage model that can effectively simulate the behavior of RC structural members with chloride ions attacks.

2. To investigate the mechanical behavior of RC structural members under various degrees of rebar corrosion through experimental work.
3. To predict the future strength of RC structural members for a different exposition periods by using the proposed model.

4. To confirm the importance of orthotropic analysis in structural performance of RC structural members.

5. To examine the dead load effects to diffusion process of chloride ions in real concrete bridge girders analysis.

1.4 Research scope and limitations

In this study, the effects of rebar corrosion were considered which consist of reduction of effective cross section of rebar and initiation of volume expansion pressure to the concrete surface. However, the adhesion or bond factor between concrete and steel surface is not taken into account entirely.

Two kind of yield criterion were applied in this study, which are von Mises and Drucker Prager criterion and they are adopted to simulate steel and concrete material, respectively.

While rebar corrosion is mainly initiated from chloride ions attacks and carbonation process, this study considered the rebar corrosion due to chloride ions attacks only. The carbonation factor will be recommend to conduct in the future.

In investigating the factor of structure’s scale to the dead load effect, the design of the structure in 15 and 20 meters length was taking into consideration of the dead load only. The design load is neglected.
1.5 Organization of the thesis

The dissertation is organized into six chapters. As introduction, Chapter I introduces an overview on the deterioration of concrete structures which are mainly caused by chloride ions attacks.

In Chapter II, the mechanical models of concrete and steel reinforcement bar are discussed in details. The derivation of pressure independent of elastic-plastic constitutive equation under von Mises criterion for concrete material is obtained. In other hand, the pressure dependent yield criterion for steel reinforcement bar is derived based on the Drucker-Prager plasticity model.

Next, an overview of continuum damage mechanics is presented prior to the introducing the mechanical and chemical damage variables. Then, an analysis procedure to assess the damage caused by the chemical and mechanical loading is proposed at the end of the chapter.

Next, the experimental work is reviewed in Chapter III and the procedures of electrolytic corrosion method and the static loading test is briefly presented. Then, results of the experimental testing are revealed.

In Chapter IV, the chemical-mechanical damage analysis is simulated. First, the experimental results were validated by the numerical analysis. Next, the damage analysis was furthered by predicting the future strength of the structure under various exposition periods. Then, the importance of orthotropic analysis instead of isotropic was discussed. The discussion is continued in determining the influence of marine environment on structural performance under both analysis conditions. Finally, the effect of dead load and the hydrostatic stress to the diffusion process is highlighted at the end of the study.

The findings along this research are summarized and discussed in Chapter V. The conclusions are described and recommendations for future work are presented.
CHAPTER II

MECHANICAL MODEL AND THE DAMAGE ASSESSMENT OF RC STRUCTURAL MEMBERS

2.1 Introduction

In this study, the pressure independent yield criterion has been adopted to simulate the behavior of reinforcing steel bars. Next, the pressure dependent yield criterion is derived for concrete material based on the plasticity model.

In addition, the deterioration of RC structure is highly influenced by chemical attack from the evolution of chloride ions penetration. Apart of it, the impact of external load causes cracking of concrete and exposes steel reinforcement surface to the chloride ion’s threat. The direct exposure through the cracking development increases the possibility of rebar corrosion under the sufficient oxygen and humidity. Then, the assessment of damage from chemical and mechanical attacks is investigated and the analysis procedure is developed in order to evaluate the residual strength of RC structure.
2.2 Pressure Independent of Elastic-Plastic Constitutive Equation

2.2.2 Constitutive model

In this study, the mathematical description of the material behavior is presented and leads the relation between the stress and strain tensor in a material point of the body, plays an important role to simulate the structural member behavior. Most constitutive models based are on the theories of elasticity and plasticity. In order to simulate the strength of materials in various states, failure criteria defined with stress invariants were developed in 1-parameter criterion and 2-parameter criterion. Many researchers such as Bresler et al., William et al., Ottosen or Hsieh et al. have attempted to extend the application of plasticity model until 3 to 5 parameters criterion. However, this study considered 1 and 2 parameters criterion to describe the deformation characteristics of reinforced concrete in the ultimate stress state.

2.2.3 J2 theory

When the second invariant of deviatoric stress tensor, $J_{2D}$ reaches a certain value, yielding or plastic domain will take place. Von Mises in year 1913 called this criterion as distortional energy theory or shearing-stress criteria, in which when distortional energy reaches a value which equals to the distortional energy at yield in simple tension, the yielding is occurred. The Von Mises yield surfaces in principal stress coordinate circumscribes a cylinder with certain radius around the hydrostatic axis as shown in Figure 1. By this criterion, the effect of hydrostatic pressure on compression domain is neglected. This phenomenon is reasonable for metal plasticity.
According to this criterion, the function of the von Mises model can be expressed by,

\[ f_{vm}(J_2) = J_2 - k^2 < 0 \Rightarrow \text{elastic} \]
\[ f_{vm}(J_2) = J_2 - k^2 = 0 \Rightarrow \text{yield} \]

where \( k \) is the material constant and the second invariant of deviatoric stress, \( J_{2D} \) can be illustrated as follow;

\[ J_{2D} = \frac{1}{2} \sigma'_{ij} \sigma'_{ij} - k^2 \]

And the deviatoric stress, can be written as

\[ \sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \]

\[ f(J_2) = \frac{1}{6} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \right] - k^2 \]

The size of yield surface expands uniformly is governed by the value of \( k^2 \), which depends upon plastic strain history. For von Mises criterion, the constant value, \( k^2 \) can be
expressed as in Eq. 2.5 by defining $\sigma_y = \sigma_{eqs}$ when the yielding occurred in one-dimensional tensile test as shown in Figure 2.

$$k^2 = \frac{1}{3} \sigma_y^2$$ \hspace{1cm} 2.5

![Figure 2: Uniaxial tensile test of steel bar](image)

Substitute Eq 2.5 into 2.1, the function becomes

$$f_{vm} (J_2, e_{eqs}^p) = J_{2D} - \frac{1}{3} \sigma_{eqs}^2 = 0$$ \hspace{1cm} 2.6

and its differentiation

$$\frac{\partial f_{vm}}{\partial \sigma_{eqs}} = \left(\frac{\partial f_{vm}}{\partial J_{2D}}\right) \left(\frac{\partial J_{2D}}{\partial \sigma_{eqs}}\right) = \frac{2}{3} \sigma_{eqs}$$ \hspace{1cm} 2.7
2.2.4 Elastic-plastic constitutive equation of pressure independent (von Mises criteria)

Details of steps of deriving the incremental stress-strain for the VM yield criteria are given below. Assuming that the total strain increment $d\varepsilon_{ij}$ as in Figure 3 can be decomposed into incremental of elastic strain, $d\epsilon_{ij}^e$ and incremental of plastic strain, $d\epsilon_{ij}^p$ components to yield

$$d\varepsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p \quad 2.8$$

Generally, the incremental stress can be expressed as

$$d\sigma_{ij} = D_{ijkl}d\varepsilon_{kl} \quad 2.9$$

In plastic loading, both initial yield and subsequent stress states must satisfy the yield condition

$$f_{VM}(\sigma_{ij}, \varepsilon_{ij}^p) = 0 \quad 2.10$$

Continuous yield condition,

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p = 0 \quad 2.11$$

Incremental of plastic strain is calculated using flow rule associated with the yield criterion $f$

$$d\epsilon_{ij}^p = \lambda_{vm} \frac{\partial f_{VM}}{\partial \sigma_{ij}} \quad 2.12$$
Figure 3: Typical stress-strain curve

Figure 3 shows, if the equivalent plastic strains increase, then the equivalent stress also increases. By utilizing the theory of plastic work and work hardening, we can define the equivalent plastic strain increment in terms of the plastic work per unit volume in the form

\[ dW^p = \sigma_{eqs} d\varepsilon_{eqs} \]  \hspace{1cm} (2.13)

In this study, the incremental of stress for VM is incorporated with hardening rule, where the hardening coefficient parameter is defined by the ratio of equivalent stress increment, \( d\sigma_{eqs} \) to the increment of equivalent plastic strain, \( d\varepsilon_{eqs} \).

\[ H' = \frac{d\sigma_{eqs}}{d\varepsilon_{eqs}} \]  \hspace{1cm} (2.14)

If we consider the incremental stress for the linear elastic path. Substitute Eq. 2.8 in Eq. 2.9, the incremental of elastic strain can be separated into

\[ d\sigma_{ij} = D_{ijkl}^e d\varepsilon_{kl} = D_{ijkl}^e (d\varepsilon_{kl} - d\varepsilon_{eqs}) \]  \hspace{1cm} (2.15)

And fourth tensor of elastic stiffness can be expressed by Lame constant as;
\[ D_{ijkl}^e = \lambda (\delta_{ij} \delta_{kl}) + (\mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})) \quad 2.16 \]

By using equation above 2.10, when yielding occurs, \( df = 0 \). Then insert hardening, \( H' \) in Eq 2.14

\[ df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \sigma_{eq}} \frac{\partial \sigma_{eq}}{\partial \varepsilon^p} d\varepsilon^p = 0 \quad 2.17 \]

From 2.7, the associated flow rule from Eq 2.12 can be expressed as

\[ d\varepsilon^p = \lambda \frac{\partial f}{\partial \sigma_{eq}} = \frac{2}{3} \sigma_{eq} \lambda \quad 2.18 \]

Substitute Eq 2.18 into 2.17

\[ \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = - \frac{\partial f}{\partial \sigma_{eq}} \frac{\partial \sigma_{eq}}{\partial \varepsilon^p} d\varepsilon^p = \frac{4}{9} H' \sigma_{eq}^2 \lambda \quad 2.19 \]

From Eq 2.8, multiply both side with \( D_{ijkl} \)

\[ D_{ijkl} d\varepsilon_{kl} = D_{ijkl} d\varepsilon_{kl}^e - D_{ijkl} d\varepsilon_{kl}^p \quad 2.20 \]

From Eq 2.9, total strain increment can be expressed by

\[ D_{ijkl} d\varepsilon_{kl} = d\sigma_{ij} + D_{ijkl} d\varepsilon_{kl}^p \quad 2.21 \]

Multiply both side with \[ \frac{\partial f}{\partial \sigma_{ij}} \]

\[ \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} d\varepsilon_{kl} = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} d\varepsilon_{kl}^p \quad 2.22 \]
\[
\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} d\epsilon_{kl} = \frac{4}{9} H' a^2 \lambda + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}(\lambda \frac{\partial f}{\partial \sigma_{ij}}) \tag{2.23}
\]

And we can express \( \lambda \) as

\[
\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} d\epsilon_{kl}}{\frac{4}{9} H' a^2 + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial f}{\partial \sigma_{ij}}} \tag{2.24}
\]

Then, recall the form of stress increment in Eq 2.15. Substitute Eq 2.12 and Eq 2.24 into it, then we can get the stress increment as below

\[
d\sigma_{ij} = D_{ijkl}^e d\epsilon_{kl} - D_{ijkl} \left[ \frac{\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} d\epsilon_{kl}}{\frac{4}{9} H' a^2 + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial f}{\partial \sigma_{ij}}} \right] \frac{\partial f}{\partial \sigma_{ij}} \tag{2.25}
\]

Rearrange Eq. 2.25 and substitute 2.19 into it and change the index notation to obtain

\[
d\sigma_{ij} = \left[ D_{ijkl}^e - \frac{\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}^e}{\frac{4}{9} H' a^2 + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}^e} \frac{\partial f}{\partial \sigma_{ij}} \right] d\epsilon_{kl} \tag{2.26}
\]

Here in, each term involved can be simplified as follows,

\[
\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}} \left( \frac{1}{2} \sigma'_{kl} \sigma'_{kl} - k^2 \right) = \frac{\partial}{\partial \sigma'_{mn}} \left( \frac{1}{2} \sigma'_{kl} \sigma'_{kl} - k^2 \right) \frac{\partial \sigma'_{mn}}{\partial \sigma_{ij}} = (\sigma'_{kl} \delta_{km} \delta_{ln}) \frac{\partial f}{\partial \sigma_{ij}} (\sigma_{mn} - \frac{1}{3} \sigma_{se} \delta_{mn}) \tag{2.27}
\]
\[ \sigma'_{mn}(\delta_{im}\delta_{jn} - \frac{1}{3}\delta_{ij}\delta_{mn}) \]
\[ = \sigma'_{ij} - \frac{1}{3}\sigma'_{mm}\delta_{ij} \]
\[ = \sigma'_{ij} \]

Else, the isotropic-elastic body stiffness as

\[ D_{ijkl}^e = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ij}\delta_{jl} + \delta_{il}\delta_{jk}) \]
\[ \frac{\partial f}{\partial \sigma_{ij}} = \left\{ \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right\} \sigma'_{ij} \]
\[ = \lambda \sigma'_{ii}\delta_{kl} + 2\mu \sigma'_{kl} \]
\[ = 2\mu \sigma'_{kl} \]

Substitute 2.27 into 2.26 to simplify the derivation

\[ d\sigma_{ij} = \left[ D_{ijkl}^e - \frac{D_{ijrs}^e \sigma'_{rs} D_{mnkl}^e \sigma'_{mn}}{\frac{4}{9} H' \sigma_{eq}^2 + \sigma'_{ab} D_{abcd}^e \sigma'_{cd}} \right] d\varepsilon_{kl} \]
\[ 2.29 \]

Substitute 2.28 in 2.29, then we can write

\[ d\sigma_{ij} = \left[ D_{ijkl}^e - \frac{2\mu \sigma'_{ij} 2\mu \sigma'_{kl}}{\frac{4}{9} H' \sigma_{eq}^2 + 2\mu \sigma'_{ab} \sigma'_{cd}} \right] \delta_{lk} \delta_{jl} d\varepsilon_{kl} \]
\[ 2.30 \]

Finally, solve Eq 2.30 by using the notation \( \sigma'_{ij} \sigma'_{ij} = \frac{2}{3} \sigma_{eq}^2 \), thus the incremental of stress for VM used in this study can be simplified as below

\[ d\sigma_{ij} = \left[ D_{ijkl}^e - \frac{4 \mu^2 \sigma'_{ij} 2\mu \sigma'_{kl}}{\frac{4}{9} H' \sigma_{eq}^2 + 2\mu \sigma'_{ab} \sigma'_{cd}} \right] \delta_{lk} \delta_{jl} d\varepsilon_{kl} \]
\[ 2.31 \]
\[ d\sigma_{ij} = \left[ D_{ijkl} - \frac{\mu^2 \sigma'_{ij} \sigma'_{kl}}{H' \sigma^2_{eq} + \mu \frac{1}{3} \sigma^2_{eqs}} \right] \delta_{ik} \delta_{jl} d\varepsilon_{kl} \]  

2.32

And the matrix form of \([D]^p\) can be formed by

\[
[D]^p = \frac{9\mu^2}{\sigma_{eq}^2 (H' + 3\mu)} \begin{bmatrix}
\sigma_{11}^2 & \sigma_{12}^2 & \sigma_{13}^2 & \sigma_{14}^2 & \sigma_{15}^2 & \sigma_{16}^2 \\
\sigma_{21}^2 & \sigma_{22}^2 & \sigma_{23}^2 & \sigma_{24}^2 & \sigma_{25}^2 & \sigma_{26}^2 \\
\sigma_{31}^2 & \sigma_{32}^2 & \sigma_{33}^2 & \sigma_{34}^2 & \sigma_{35}^2 & \sigma_{36}^2 \\
\sigma_{41}^2 & \sigma_{42}^2 & \sigma_{43}^2 & \sigma_{44}^2 & \sigma_{45}^2 & \sigma_{46}^2 \\
\sigma_{51}^2 & \sigma_{52}^2 & \sigma_{53}^2 & \sigma_{54}^2 & \sigma_{55}^2 & \sigma_{56}^2 \\
\sigma_{61}^2 & \sigma_{62}^2 & \sigma_{63}^2 & \sigma_{64}^2 & \sigma_{65}^2 & \sigma_{66}^2 \\
\end{bmatrix}
\]

2.34

Symmetry

2.3 Pressure Dependent Constitutive Equation

2.3.2 Elasto-plastic constitutive equation of pressure dependent (Drucker Prager criteria)

In two parameter model, the yielding criterion of concrete combines the effect of hydrostatic pressure in compression zone and fracture property in tension zone. In other words, this is an extended of von Mises yield criterion which includes the effect of hydrostatic pressure on the shearing resistance of material. The graphic representation of linear Drucker-Prager yield surfaces is presented in Figure 4 in the principal space. From the figure, the phenomenological explanation for the pressure dependent flow due to internal friction is provided, which is a typical feature for brittle materials.
The Drucker-Prager yield criterion can be described as:

\[
f_{DP}(J_2, I_1) = \sqrt{J_2} + I_1 \alpha - k < 0; \text{ Elastic} \tag{2.35 (a)}
\]

\[
f_{DP}(J_2, I_1) = \sqrt{J_2} + I_1 \alpha = 0; \text{ Plastic} \tag{2.35 (b)}
\]

Once the stress reaches on the yield surface, the plastic condition satisfied Eq. 2.35 chloride ions and can be simplified as stated:

\[
df_{DP} = 0 \tag{2.36 (a)}
\]

It follows with chain rule,

\[
df_{DP} = \frac{\partial f_{DP}}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \tag{2.36 (b)}
\]

By noting that the second invariant of deviatoric stress, \(J_{2D}\) and \(I_1 = \sigma_{ij} \delta_{ij}\), we can derive the function in Eq. 2.35 as stated:

\[
\frac{\partial f_{DP}}{\partial \sigma_{ij}} = \left( \frac{\partial f_{DP}}{\partial \sqrt{J_2}} \frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} \right) + \left( \frac{\partial f_{DP}}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{ij}} \right) = \left( \frac{J_{2D}^{1/2}}{2} \sigma'_{ij} \right) + (\alpha \delta_{ij}) = \frac{\sigma'_{ij}}{2\sqrt{J_{2D}}} + \alpha \delta_{ij} \tag{2.37}
\]
The incremental of plastic strain is calculated by using flow rule. Substitute Eq. 2.37 into 2.12, it new equation is written as:

\[ d\varepsilon_{ij}^p = \lambda_{DP} \frac{\partial f_{DP}}{\partial \sigma_{ij}} = \lambda_{DP} \left( \frac{\sigma_{ij}'}{2\sqrt{J_2D}} + a\delta_{ij} \right) \]  

where \( \lambda_{DP} \) is plastic multiplier of Drucker-Prager model. Then, the general increment stress relationship can be written by separating the total strain increment into elastic and plastic component as below:

\[ d\sigma_{ij} = D_{ijkl} (d\varepsilon_{ij} - d\varepsilon_{ij}^p) \]  

and fourth order tensor of elastic stiffness can be expressed by Lame constant as:

\[ D_{ijkl}^e = \lambda (\delta_{ij}\delta_{kl}) + \mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \]  

Under a uniaxial state of stress, the yield stress \( \sigma_y \) is \( \sigma_y > 0 \) (Fig. 5), \( f=0 \) so that \( k \) can be expressed as

![Diagram](image.png)

Figure 5 : Determination of k value using J2D-I plot

18
\[ k = \left[ \alpha + \frac{1}{\sqrt{3}} \right] \sigma_y \]  

2.41

Substitute Eq. 2.41 into 2.35, the effective stress \( \sigma_{e qs} = \sigma_y \) can be defined as;

\[ \sqrt{J_{2D}} + l_1 \alpha = \left[ \alpha + \frac{1}{\sqrt{3}} \right] \sigma_{e qs} \]  

2.42 (a)

\[ \sigma_{e qs} = \frac{\sqrt{J_{2D}} + l_1 \alpha}{\left[ \alpha + \frac{1}{\sqrt{3}} \right]} \]  

2.42 (b)

\[ \sigma_{e qs} \left[ \alpha + \frac{1}{\sqrt{3}} \right] = \sqrt{J_{2D}} + l_1 \alpha \]  

2.42 (c)

The effective plastic strain \( d\varepsilon_p \) can be calculated by plastic work,

\[ dW^p = \sigma_{ij} d\varepsilon_{ij}^p = \sigma_{e qs} d\varepsilon_p \]  

2.43

Substitute Eq 2.38 into 2.43 gives

\[ \sigma_{ij} \lambda_{DP} \left( \frac{\sigma_{ij}'}{2\sqrt{J_{2D}}} + \alpha \delta_{ij} \right) = \sigma_{e qs} d\varepsilon_p \]  

2.44

Since \( \sigma_{ij}' \sigma_{ij}' = 2J_{2D} \) and \( \sigma_{ij}' \delta_{ij} = l_1 \), expand Eq 2.44 to form,

\[ \lambda_{DP} \left( \frac{\sigma_{ij}' \sigma_{ij}'}{2\sqrt{J_{2D}}} + \alpha \sigma_{ij}' \delta_{ij} \right) = \sigma_{e qs} d\varepsilon_p \]  

2.45

\[ \lambda_{DP} \left( \frac{2J_{2D}}{2\sqrt{J_{2D}}} + l_1 \alpha \right) = \sigma_{e qs} d\varepsilon_p \text{ or } \lambda_{DP} \left( \sqrt{J_{2D}} + l_1 \alpha \right) = \sigma_{e qs} d\varepsilon_p \]  

2.46

Substitute Eq. 2.42 (c) into 2.46,
\[ \lambda_{DP} \sigma_{eqs} \left[ \alpha + \frac{1}{\sqrt{3}} \right] = \sigma_{eqs} \varepsilon_p \]  

2.47

So that,

\[ \varepsilon_p = \lambda_{DP} \left[ \alpha + \frac{1}{\sqrt{3}} \right] \]  

2.48

By squaring Eq. 2.38, the plastic multiplier can be factorize as,

\[ d\varepsilon_p^p \cdot d\varepsilon_i^p = \lambda_{DP}^2 \left( \frac{\sigma'_{ij} + \alpha \delta_{ij}}{2\sqrt{I_{2D}}} \right) \left( \frac{\sigma'_{ij} + \alpha \delta_{ij}}{2\sqrt{I_{2D}}} + \alpha \delta_{ij} \right) \]  

2.49

Noting that

\[ \sigma'_{ij} \cdot \delta_{ij} = \sigma_{ii} = 0, \quad \delta_{ij}, \delta_{ij} = 3 \]  

2.50

Expand Eq. 2.49 and solve it

\[ d\varepsilon_p^p \cdot d\varepsilon_i^p = \lambda_{DP}^2 \left( \frac{\sigma'_{ij}^2 \sigma'_{ij} + \sigma_{ij}^2 \sigma_{ij}}{2\sqrt{I_{2D}}} + \alpha \frac{\sigma'_{ij} \cdot \delta_{ij}}{2\sqrt{I_{2D}}} + \alpha^2 \delta_{ij} \delta_{ij} \right) \]  

2.51 (a)

\[ = \lambda_{DP}^2 \left( \frac{2I_{2D}}{\alpha} + \alpha^2 3 \right) = \lambda_{DP}^2 \left( \frac{1}{2} + 3 \alpha \right) \]  

2.51 (b)

Hence,

\[ \lambda_{DP} = \sqrt{\frac{d\varepsilon_p^p \cdot d\varepsilon_i^p}{\left( \frac{1}{2} + 3 \alpha \right)}} \]  

2.52
Substitute Eq. 2.52 into 2.48, the \( d\varepsilon_p \) can be derive as,

\[
d\varepsilon_p = \frac{d\varepsilon_i^p d\varepsilon_j^p}{\sqrt{\left(\frac{1}{2} + 3\alpha^2\right)}} \left[ \alpha + \frac{1}{\sqrt{3}} \right] \tag{2.53}
\]

In plastic loading, the yield and subsequent stress state must satisfy the condition;

\[
f_{DP} (\sigma_{ij}, \varepsilon_{ij}^p, k(\varepsilon_p)) = 0
\]

whereas the yield criterion of Drucker-Prager is a function of stress \( \sigma_{ij} \), plastic strain \( \varepsilon_{ij}^p \) and \( k(\varepsilon_p) \). So that, plastic flow is governed by the consistency condition, by implying that,

\[
df_{DP} = \frac{\partial f_{DP}}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f_{DP}}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p + \frac{\partial f_{DP}}{\partial k(\varepsilon_p)} dk(\varepsilon_p) = 0 \tag{2.54}
\]

Then, substitute Eq. 2.38, 2.39 and 2.53 into Eq. 2.54, we can get the consistency condition as;

\[
\frac{\partial f_{DP}}{\partial \sigma_{ij}} \left[ D_{ijkl}^p \left( d\varepsilon_{kl} - \left( \lambda_{DP} \frac{\partial f_{DP}}{\partial \sigma_{ij}} \right) \right) \right] + \frac{\partial f_{DP}}{\partial \varepsilon_{ij}^p} \left( \lambda_{DP} \frac{\partial f_{DP}}{\partial \sigma_{ij}} \right) + \frac{\partial f_{DP}}{\partial k(\varepsilon_p)} \left( \frac{d\varepsilon_{ij}^p d\varepsilon_{ij}^p}{\sqrt{\left(\frac{1}{2} + 3\alpha^2\right)}} \left[ \alpha + \frac{1}{\sqrt{3}} \right] \right) = 0 \tag{2.55}
\]

Expand Eq. 2.55 and substitute Eq. 2.52 into it, the equation yield to

\[
\frac{\partial f_{DP}}{\partial \sigma_{ij}} D_{ijkl}^p d\varepsilon_{kl} - \lambda_{DP} \frac{\partial f_{DP}}{\partial \sigma_{ij}} D_{ijkl}^p d\varepsilon_{ij}^p + \lambda_{DP} \frac{\partial f_{DP}}{\partial \varepsilon_{ij}^p} \left( \frac{\partial f_{DP}}{\partial \sigma_{ij}} \right) + \left( \lambda_{DP} \frac{\partial f_{DP}}{\partial k(\varepsilon_p)} \left[ \alpha + \frac{1}{\sqrt{3}} \right] \right) = 0
\]
Setting Eq. 2.56 to get the plastic multiplier, \( \lambda_{DP} \)

\[
\frac{\partial f_{DP}}{\partial \sigma_{ij}} D_{ijkl}^e d_{kl} = \lambda_{DP} \left( \frac{\partial f_{DP}}{\partial \sigma_{ab}} \frac{D_{e}}{D_{abmn}} \frac{\partial f_{DP}}{\partial \sigma_{mn}} - \frac{\partial f_{DP}}{\partial \varepsilon_{ab}^p} \frac{\partial f_{DP}}{\partial \sigma_{ab}} - \frac{\partial f_{DP}}{\partial \varepsilon_{p}} \left[ \alpha + \frac{1}{\sqrt{3}} \right] \right) \quad 2.57
\]

\[
\lambda_{DP} = \frac{\frac{\partial f_{DP}}{\partial \sigma_{ij}} D_{ijkl}^e d_{kl}}{\frac{\partial f_{DP}}{\partial \sigma_{ab}} \frac{D_{e}}{D_{abmn}} \frac{\partial f_{DP}}{\partial \sigma_{mn}} - \frac{\partial f_{DP}}{\partial \varepsilon_{ab}^p} \frac{\partial f_{DP}}{\partial \sigma_{ab}} - \frac{\partial f_{DP}}{\partial \varepsilon_{p}} \left[ \alpha + \frac{1}{\sqrt{3}} \right] } \quad 2.58
\]

The general form of plastic multiplier is stated in Eq. 2.58. Considering the hardening effect to the Drucker-Prager constitutive equation, the ratio between the equivalent of stress and the equivalent plastic strain can be expressed as the hardening modulus,

\[
H = \frac{\partial \sigma_{eq}}{\partial \varepsilon_{p}} \quad 2.59
\]

The general form of Drucker-Prager function during plastic loading can be stated as;

\[
f(J_{2D}, I_1, \varepsilon_{p}) = J_{2D} + I_1 \alpha - k^2 = J_{2D} + I_1 \alpha - \left[ \alpha^2 + \frac{1}{3} \right] \sigma_{eq}^2 \quad 2.60
\]

The differentiation of Eq. 2.60 to \( \sigma_{eqs} \) yields

\[
\frac{\partial f_{DP}}{\partial \sigma_{eq}} = - \left[ 2 \alpha^2 + \frac{2}{3} \right] \sigma_{eq} \quad 2.61
\]

From Eq. 2.59, the derivation of DP function to the incremental plastic strain \( d \varepsilon_{p} \) can be written as

\[
\frac{\partial f_{DP}}{\partial \varepsilon_{p}} = \frac{\partial f_{DP}}{\partial \sigma_{eq}} \frac{\partial \sigma_{eq}}{\partial \varepsilon_{p}} = \frac{\partial f_{DP}}{\partial \sigma_{eq}} (H) \quad 2.62
\]
Then, substitution of Eq. 2.61 into Eq. 2.62 gives

$$\frac{\partial f_{DP}}{\partial \varepsilon_p} = - \left[ 2\alpha^2 + \frac{2}{3} \right] \sigma_{eq}(H)$$  \hspace{1cm} 2.63

In Eq. 2.35 and Eq. 2.60, the DP function does not depend on $\varepsilon_{ij}^p$. So, in DP derivation, the term of $\frac{\partial f}{\partial \varepsilon_{ij}}$ can be eliminated in general form of plastic multiplier as shown in Eq. 2.58. By substituting Eq. 2.37, 2.40 and 2.69 into Eq. 2.58, the deviation turns into

$$\lambda_{DP} = \frac{\left( \frac{\sigma'_{ij}}{\sqrt{2}J_{2D}} + \alpha \delta_{ij} \right) \lambda (\delta_{ij} \delta_{kl}) + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) d\varepsilon_{kl}}{\left( \frac{\sigma'_{ab}}{\sqrt{2}J_{2D}} + \alpha \delta_{ab} \right) \lambda (\delta_{ab} \delta_{mn}) + \mu (\delta_{am} \delta_{bn} + \delta_{an} \delta_{bm}) \left( \frac{\sigma'_{mn}}{\sqrt{2}J_{2D}} + \alpha \delta_{mn} \right) + \left[ 2\alpha^2 + \frac{2}{3} \right] \sigma_{eq}(H) \left[ \alpha + \frac{1}{\sqrt{3}} \right]}$$

Using the notation from Eq. 2.50, Eq. 2.64 can be solve as

$$\lambda_{DP} = \left[ \frac{\frac{\mu \sigma'_{kl}}{\sqrt{2}J_{2D}} + \alpha (3\lambda + 2\mu) \delta_{kl}}{3\alpha^2 (3\lambda + 2\mu) + \mu + \left[ 2\alpha^2 + \frac{2}{3} \right] \sigma_{eq}(H) \left[ \alpha + \frac{1}{\sqrt{3}} \right]} \right] d\varepsilon_{kl} \hspace{1cm} 2.65$$

Next, reveal Eq. 2.39 and substitute Eq. 2.38 into it generate

$$d\sigma_{ij} = D_{ijkl}^p (d\varepsilon_{kl} - d\varepsilon_{kl}^p) = D_{ijkl}^p d\varepsilon_{kl} - D_{ijkl}^p \lambda_{DP} \frac{\partial f_{DP}}{\partial \sigma_{ij}}$$
And substitute again the derivation in Eq. 2.64 into 2.66 to form

\[ d\sigma_{ij} = D_{ijkl}^e d\epsilon_{kl} - D_{ijkl} \left[ \frac{\mu\sigma_{kl}'}{\sqrt{J_2D}} + \alpha(3\lambda + 2\mu)\delta_{kl} \right] \frac{d\epsilon_{kl}}{3\alpha^2(3\lambda + 2\mu) + \mu + \left[ 2\alpha^2 + \frac{2}{3} \right] \sigma_{eq}(H) \left[ \alpha + \frac{1}{\sqrt{3}} \right]} d\epsilon_{kl} \left( \frac{\sigma_{ij}'}{2\sqrt{J_2D}} + \alpha\delta_{ij} \right) \]

2.67

Substitute \( D_{ijkl}^p \) into Eq. 2.40. Then, expand the \([D]^p\) in Eq. 2.67 and modify the index notation,

\[ [D]^p = \lambda(\delta_{ij}\delta_{kl}) + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \frac{\mu\sigma_{kl}'}{\sqrt{J_2D}} + \alpha\delta_{kl} \frac{d\epsilon_{kl}}{3\alpha^2(3\lambda + 2\mu) + \mu + \left[ 2\alpha^2 + \frac{2}{3} \right] \sigma_{eq}(H) \left[ \alpha + \frac{1}{\sqrt{3}} \right]} \]

2.68

The final constitutive equation of DP can be expressed as,

\[ d\sigma_{ij} = D_{ijkl}^e d\epsilon_{kl} - \left[ \frac{\mu\sigma_{kl}'}{\sqrt{J_2D}} + \alpha(3\lambda + 2\mu)\delta_{kl} \right] \frac{d\epsilon_{kl}}{3\alpha^2(3\lambda + 2\mu) + \mu + \left[ 2\alpha^2 + \frac{2}{3} \right] \sigma_{eq}(H) \left[ \alpha + \frac{1}{\sqrt{3}} \right]} \delta_{mk}\delta_{nl} d\epsilon_{kl} \]

2.69

The plastic stiffness matrix form of \([D]^p\) for DP model can be expressed as below

\[
[D]^p = \begin{bmatrix}
[H^{e} + H' \sigma_{11}^e] & \cdots & [H^{e} + H' \sigma_{11}^e] & \cdots & [H^{e} + H' \sigma_{11}^e] & \cdots & [H^{e} + H' \sigma_{11}^e] & \cdots & [H^{e} + H' \sigma_{11}^e] \\
[H^{e} + H' \sigma_{12}^e] & \cdots & [H^{e} + H' \sigma_{12}^e] & \cdots & [H^{e} + H' \sigma_{12}^e] & \cdots & [H^{e} + H' \sigma_{12}^e] & \cdots & [H^{e} + H' \sigma_{12}^e] \\
[H^{e} + H' \sigma_{13}^e] & \cdots & [H^{e} + H' \sigma_{13}^e] & \cdots & [H^{e} + H' \sigma_{13}^e] & \cdots & [H^{e} + H' \sigma_{13}^e] & \cdots & [H^{e} + H' \sigma_{13}^e] \\
[H^{e} + H' \sigma_{21}^e] & \cdots & [H^{e} + H' \sigma_{21}^e] & \cdots & [H^{e} + H' \sigma_{21}^e] & \cdots & [H^{e} + H' \sigma_{21}^e] & \cdots & [H^{e} + H' \sigma_{21}^e] \\
[H^{e} + H' \sigma_{22}^e] & \cdots & [H^{e} + H' \sigma_{22}^e] & \cdots & [H^{e} + H' \sigma_{22}^e] & \cdots & [H^{e} + H' \sigma_{22}^e] & \cdots & [H^{e} + H' \sigma_{22}^e] \\
[H^{e} + H' \sigma_{23}^e] & \cdots & [H^{e} + H' \sigma_{23}^e] & \cdots & [H^{e} + H' \sigma_{23}^e] & \cdots & [H^{e} + H' \sigma_{23}^e] & \cdots & [H^{e} + H' \sigma_{23}^e] \\
[H^{e} + H' \sigma_{31}^e] & \cdots & [H^{e} + H' \sigma_{31}^e] & \cdots & [H^{e} + H' \sigma_{31}^e] & \cdots & [H^{e} + H' \sigma_{31}^e] & \cdots & [H^{e} + H' \sigma_{31}^e] \\
[H^{e} + H' \sigma_{32}^e] & \cdots & [H^{e} + H' \sigma_{32}^e] & \cdots & [H^{e} + H' \sigma_{32}^e] & \cdots & [H^{e} + H' \sigma_{32}^e] & \cdots & [H^{e} + H' \sigma_{32}^e] \\
[H^{e} + H' \sigma_{33}^e] & \cdots & [H^{e} + H' \sigma_{33}^e] & \cdots & [H^{e} + H' \sigma_{33}^e] & \cdots & [H^{e} + H' \sigma_{33}^e] & \cdots & [H^{e} + H' \sigma_{33}^e]
\end{bmatrix}
\]

symmetry

2.70
REFERENCES


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