Operational matrix based on Genocchi polynomials for solution of delay differential equations

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Abstract

In this paper, we present a new simple and effective algorithm for solving generalized Pantograph equations, delay differential equations with neutral terms and delay differential system with constant and variable coefficients. The new method is based on one of the Appell polynomials, namely Genocchi polynomials. We first introduce the properties of Genocchi polynomials and employed them to construct the operational matrices of derivative. Collocation method based on this operational matrix is used. Error estimate for this scheme based on the Generalized pantograph equations is reproduced. Only few terms of Genocchi polynomials are needed to obtain very good results. Numerical examples with comparison show the simplicity, efficiency and accuracy of the method.

1. Introduction

Delay differential equations play an important role in explaining different phenomena in many different fields of study such as biology, physics, economics, electrodynamics, control theory etc. [1–3]. According to [2], the name pantograph refers to the device that collect electric current from overhead lines for electric trains or trams. The clear picture of this device is shown in [2]. The pantograph equations was originated from the work of Ockendon and Tayler [4] in which the system of collecting overhead electricity for trains is redesigned and modeled to ensure the contact is maintain throughout. Pantograph equations are fundamental when a phenomenon or a process fail to be modeled by the ordinary differential equations. In recent years many researchers have focus in the numerical treatment of pantograph equations. Tohidi et al. in [5] proposed a new collocation scheme based on Bernoulli operational matrix for numerical solution of generalized pantograph equation. Yusufoglu [6] proposed an efficient algorithm for solving generalized pantograph equations with linear functional argument. In [7], Yang and Huang presented a spectral-collocation method for fractional pantograph delay-integrodifferential equations and in [8], Yüzbaşı and Mehmet presented an exponential approximation for solutions of generalized pantograph delay differential equations. Multiquadric approximation scheme was used in [9] for the numerical solution of delay differential systems of neutral type. Taylor method was used in [10] for numerical solution of generalized pantograph equations with linear functional argument. Chebyshev polynomials and Bessel polynomials are respectively used in [11,12] to obtain the solutions of generalized pantograph equations, while Adomian decomposition method and Variational iteration method are applied in [13,14] respectively for the solution of delay differential equations. In this paper, an important member of Appell polynomials called the Genocchi polynomials is used. Though Genocchi polynomials are not based on orthogonal functions but they possess operational matrix of derivative and when it comes to function approximation, this polynomials share with other members of the Appell family, such as Bernoulli polynomials, some sound and advantageous properties over other classical orthogonal polynomials such as Legendre polynomials, Chebyshev polynomials, Laguerre polynomials and etc. These advantages are stated in [5].

Motivated by these advantages, we used Genocchi polynomials operational matrix of derivative through collocation method to approximate the solution of delay differential equation of pantograph type and those with neutral term together with its differential systems. We based all our arguments on generalized form of pantograph equations given by [5]:

\[ y^{(m)}(t) = \sum_{i=0}^{m-1} \sum_{k=0}^{i} \beta_{i,k} y^{(i-k)}(t_{i}) + \sum_{i=0}^{m-1} \sum_{k=0}^{i} \beta_{i,k} g(t_{i}), \quad 0 \leq t \leq 1 \]
subject to the following conditions

\[ y'(0) = d_j, \quad j = 0, 1, \ldots, m - 1. \tag{2} \]

where \( a_k \) and \( \beta_k \) are real or complex coefficients, while \( p_k(t) \) and \( g(t) \) are given continuous functions in the interval \([0, 1]\).

The rest of the paper is organized as follows: Section 2, introduce some mathematical preliminaries of Genocchi polynomials. In Section 3, we apply the collocation method for solving pantograph Eq. (1) using the Genocchi operational matrix. Section 4, we show the error analysis of the proposed method. In Section 5, the proposed method is applied to several examples. Conclusion is given in Section 6.

2. Some properties of Genocchi polynomials

Genocchi numbers and polynomials have been extensively studied in many different context in branches of mathematics such as elementary number theory and complex analytic number theory, in which this polynomials are highly developed and applied, like the so called -Genocchi polynomials are developed in [15–17], their interpolation functions are also discussed in [16,18]. Genocchi polynomials are also studied in homotopy theory (stable homotopy groups of spheres), differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), and quantum physics (quantum groups). The classical Genocchi polynomials \( G_n(x) \) is usually defined by means of the exponential generating functions [19–22].

\[
\frac{2te^t}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \quad (|t| < \pi) \tag{3}
\]

where \( G_n(x) \) is the Genocchi polynomials of degree \( n \) and is given by

\[
G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k} \tag{4}
\]

The first few Genocchi polynomials are:

\[
\begin{align*}
G_0(x) &= 1 \\
G_1(x) &= x \\
G_2(x) &= 2x^2 - 1 \\
G_3(x) &= 3x^3 - 3x \\
G_4(x) &= 4x^4 - 6x^2 + 1 \\
G_5(x) &= 5x^5 - 10x^3 + 5x
\end{align*}
\]

Differentiating both sides of (4), with respect to \( x \), then we have the following as \([21,23]\)

\[
\frac{dG_n(x)}{dx} = nG_{n-1}(x), \quad n \geq 1 \tag{5}
\]

We prove one of the important property in the following Lemma

Lemma 2.1. \( G_n(1) + G_n(0) = 0, \quad n > 1 \tag{6} \)

Proof. From (3) we have that

\[
e^t = \frac{1}{2t} \left( \frac{2te^{x+1}}{e^t + 1} + \frac{2te^x}{e^t + 1} \right) = \frac{1}{2t} \sum_{n=0}^{\infty} (G_n(x+1) + G_n(x)) \frac{t^n}{n!}, \quad (|t| < \pi)
\]

i.e.

\[
\sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = \frac{1}{2t} \sum_{n=0}^{\infty} (G_n(x+1) + G_n(x)) \frac{t^n}{n!} \quad (|t| < \pi) \tag{7}
\]

from (7) we have

\[ G_n(x+1) + G_n(x) = 2nx^{n-1}, \quad n > 1. \tag{8} \]

Hence, the result follows obviously. \( \square \)

If we introduce the Genocchi vector \( G(x) \) in the form \( G(x) = [G_1(x), G_2(x), \ldots, G_n(x)] \), then the derivative of the \( G(x) \) with the aid of (5), can be expressed in the matrix form by

\[ G'(x) = MG(x) \]

where

\[
G'(x) = \begin{pmatrix}
G_1'(x) \\
G_2'(x) \\
G_3'(x) \\
\vdots \\
G_n'(x)
\end{pmatrix}, \quad M = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
2 & 0 & 0 & \cdots & 0 & 0 \\
3 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & N - 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & N
\end{pmatrix}
\]

and

\[ G(x)^T = \begin{pmatrix}
G_1(x) \\
G_2(x) \\
G_3(x) \\
\vdots \\
G_n(x)
\end{pmatrix} \]

Thus, \( M \) is \( N \times N \) operational matrix of derivative. Accordingly, the \( k \)th derivative of \( G(x) \) can be given by

\[ G^{(k)}(x) = MG^{(k-1)}(x) \Rightarrow G^{(k)}(x) = G(x)M^k \]

\[ G^{(2)}(x) = G^{(1)}(x)M = G(x)(M^2)^k \]

\[ G^{(3)}(x) = G^{(2)}(x)(M^2) = G(x)(M^3)^k \]

\[ \vdots \]

\[ G^{(k)}(x) = G(x)(M^k)^k \]

We refer to the work of E. Tohidi et al. [5] for the advantages of Appell family polynomials for approximating an arbitrary unknown function over some classical orthogonal polynomials.

3. Collocation method based on Genocchi operational matrix

In this section, we use the collocation method based on Genocchi operational matrix of derivatives to solve numerically the generalized pantograph equation. We now derive an algorithm for solving (1). To do this, let the solution of (1) be approximated by the first \( N \) terms Genocchi polynomials. Thus, we write

\[ y_N(t) \approx \sum_{n=1}^{N} c_n G_n(t) = G(t)C \tag{9} \]

where the Genocchi coefficient vector \( C \) and the Genocchi vector \( G(t) \) are given by

\[ C = [c_1, c_2, \ldots, c_N], \quad G(t) = [G_1(t), G_2(t), \ldots, G_N(t)] \]

then, the \( k \)th derivative of \( y_N(t) \) can be expressed as follows

\[ y_N^{(k)}(t) = G^{(k)}(t)C = G(t)(M^k)^k C \tag{10} \]

Substituting (9) and (10) in (1) we have

\[ G(t)(M^k)^m C = \sum_{i=0}^{l} \sum_{k=0}^{m-1} p_{i,k}(t) \alpha_{i,k} G_{i,k}(t + \beta_{i,k})(M^k)^k C + g(t) \tag{11} \]
where \( G(x_{i}t + \beta_{i}) = [G_{1}(x_{i}t + \beta_{i}), G_{2}(x_{i}t + \beta_{i}), \ldots, G_{N}(x_{i}t + \beta_{i})] \).

Also the initial condition will produce \( m \) other equations

\[
G(0)(M^{j})^{C} = d_{j}, \quad j = 0, 1, \ldots, m - 1
\]  

(12)

To find the solution \( y_{N}(x) \), we collocate (11) at the collocation points \( t_{j} = \frac{2j}{N-m} \) for \( j = 1, 2, \ldots, N - m \) to obtain

\[
G(t_{j})(M^{j})^{C} = \sum_{i=0}^{m-1} \sum_{k=0}^{N} \frac{1}{i!} \frac{d^{i}}{dt^{i}}(G_{i}(x_{j}t + \beta_{i})) C + g(t_{j})
\]  

(13)

for \( j = 1, 2, \ldots, N - m \). (13) are \( N - m \) non-linear algebraic equations. These equations together with (12) make \( N \) algebraic equations which can be solved using Newton's iterative method. Consequently \( y_{N}(x) \) given in (9) can be calculated.

4. Error analysis

Suppose that \( H = L^{2}[0,1] \) and \( \{G_{1}(t), G_{2}(t), \ldots, G_{N}(t)\} \subset H \) be the set of Genocchi polynomials and \( Y = \text{Span} \{G_{1}(t), G_{2}(t), \ldots, G_{N}(t)\} \). Also let \( f(t) \) be arbitrary element of \( H \), since \( Y \) is a finite dimensional subspace, \( f(t) \) has a unique best approximation in \( Y \), say \( f'(t) \) such that \( \forall t_{j} \in Y, \|f(t) - f'(t)\|_{2} \leq \|f(t) - y(t)\|_{2} \).

Since \( f'(t) \in Y \), then there exist the unique coefficients \( c_{1}, c_{2}, \ldots, c_{N} \) such that

\[
f(t) \approx f'(t) = \sum_{i=1}^{N} c_{i} G_{i}(t) = CG(t)
\]

where \( C = [c_{1}, c_{2}, \ldots, c_{N}] \). \( G(t) = [G_{1}(t), G_{2}(t), \ldots, G_{N}(t)] \)

In the following Lemma we prove a new way that shows how the coefficients \( c_{i} \) can be obtained.

**Lemma 4.1.** Assume that \( f \in H = L^{2}[0,1] \), is an arbitrary function approximated by the truncated Genocchi series \( \sum_{n=1}^{N} c_{n} G_{n}(x) \), then the coefficients \( c_{n} \) for \( n = 1, 2, \ldots, N \) can be calculated from the following relation

\[
c_{n} = \frac{1}{2^n} (f^{(n-1)}(1) + f^{(n-1)}(0)).
\]

Note that the power in \( (n-1) \) in \( f^{(n-1)}(x) \) denotes the \( (n-1) \)th derivative of \( f \)

**Proof.** Suppose \( f(x) \approx \sum_{n=1}^{N} c_{n} G_{n}(x) \) then, using (6) one has

\[
f(1) + f(0) = c_{1}(G_{1}(1) + G_{0}(0)) + c_{2}(G_{2}(1) + G_{2}(0)) + \cdots + c_{N}(G_{N}(1) + G_{0}(0)) = 2c_{1}
\]

\[
\Rightarrow c_{1} = \frac{1}{2} (f(1) + f(0))
\]

\[
f^{(1)}(1) + f^{(1)}(0) = 2c_{2}(G_{2}(1) + G_{1}(0)) + 3c_{3}(G_{3}(1) + G_{2}(0)) + \cdots + Nc_{N}(G_{N}(1) + G_{N}(0)) = 2c_{2}
\]

\[
\Rightarrow c_{2} = \frac{1}{2(2!)} (f^{(1)}(1) + f^{(1)}(0))
\]

\[
f^{(2)}(1) + f^{(2)}(0) = 3(2)c_{3}(G_{3}(1) + G_{2}(0)) + 4(3)c_{4}(G_{4}(1) + G_{3}(0)) + \cdots + (N(N-1))c_{N}(G_{N-1}(1) + G_{N-1}(0)) = 3c_{3}
\]

\[
\Rightarrow c_{3} = \frac{1}{2(3!)} (f^{(2)}(1) + f^{(2)}(0))
\]

Repeating this procedure i times for \( i = 1, 2, \ldots, N \), we have

\[
c_{i} = \frac{1}{2(1!)^{i}} (f^{(i-1)}(1) + f^{(i-1)}(0))
\]

this complete the proof. \( \square \)

**Theorem 4.2.** Suppose that \( f(x) \in C^{m}[0,1] \) and \( f'(t) \) is the approximated \( f(t) \) by using Genocchi polynomials, then the error bound would be obtained as follows:

\[
\|\text{error}(f(t))\|_{\infty} \leq \frac{1}{N!} G_{N} F_{N}
\]

where \( G_{N} \) and \( F_{N} \) denotes the maximum value of \( G_{N}(t) \) and \( f^{(N-1)}(t) \) \( \forall t \in [0,1] \) respectively.

**Proof.** The proof is obvious, one should consider Lemma 4.1, \( \square \)

In the following we reproduced the theorem given in [5] which gives an upper bound for the error of the considered problem (1)

**Theorem 4.3.** Let \( y(x) \) and \( y_{N}(x) \) be the exact and approximate solutions of (1) respectively. Consider (1) when \( m = 1 \) so that \( p_{1}(t) = p_{1,k}(t) \). Also assume that \( \forall t \in [0,1], \|y(t)\|_{\infty} \leq \rho, \|K_{i}(t)\|_{\infty} = \|p_{1}(t)\|_{\infty} \leq \lambda_{i}, \quad i = 1, 2, \ldots, I \) and \( \sum_{i=0}^{I} (A_{i} + E_{i}) \neq 0 ; \)

\[
\|y(t) - y_{N}(t)\|_{\infty} \leq \frac{E_{f} + \rho \sum_{i=0}^{I} E_{i}}{1 - \sum_{i=0}^{I} (A_{i} + E_{i})}
\]

where \( E_{i} = \|K_{i}(t) - K_{N}(t)\|_{\infty}, i = 1, 2, \ldots, I \) and \( E_{f} = \|f(t) - f_{N}(t)\|_{\infty} \), with \( f(t) = y(0) + \int_{0}^{t} g(\tau) d\tau \).

**Proof.** According to the assumptions above (1) becomes:

\[
y'(t) = \sum_{i=0}^{I} p_{i}(t) y(\alpha_{i} t + \beta_{i}) + g(t)
\]

Integrating both sides in the interval \( [0,t] \) and imposing initial condition we have

\[
y(t) = y(0) + \int_{0}^{t} g(\tau) d\tau + \sum_{i=0}^{I} \int_{0}^{t} p_{i}(\tau) y(\alpha_{i} \tau + \beta_{i}) d\tau
\]

putting \( z = \alpha_{i} t + \beta_{i}, \) we get,

\[
y(t) = f(t) + \sum_{i=0}^{I} \int_{\beta_{i}}^{\alpha_{i} t + \beta_{i}} \frac{1}{\alpha_{i}} p_{i}(z) y(z) dz
\]

If we assume \( K_{i}(z) = p_{i}(\frac{z - \beta_{i}}{\alpha_{i}}) \) the above equation can be written as:

\[
y(t) = f(t) + \sum_{i=0}^{I} \int_{\beta_{i}}^{\alpha_{i} t + \beta_{i}} \frac{1}{\alpha_{i}} K_{i}(z) y(z) dz
\]

Now, suppose that the functions \( K_{i}(z) \) and \( f(t) \) are expanded in terms of Genocchi polynomials, then the obtained solution \( y_{N}(x) \) is also in terms of Genocchi polynomials. Our aim here is to find an upper bound for the associated error between the exact solution \( y(x) \) and the approximated solution \( y_{N}(x) \) for Eq. (1). With considered assumptions we have;

\[
\||y(t) - y_{N}(t)||_{\infty} = \|f(t) - f_{N}(t)\| + \sum_{i=0}^{I} \int_{\beta_{i}}^{\alpha_{i} t + \beta_{i}} \frac{1}{\alpha_{i}} \|K_{i}(z) y(z)\| dz
\]

\[
- K_{N}(z) y_{N}(z) dz\|_{\infty} \leq \|f(t) - f_{N}(t)\|_{\infty} + \sum_{i=0}^{I} \int_{\beta_{i}}^{\alpha_{i} t + \beta_{i}} \frac{1}{\alpha_{i}} \||K_{i}(z) y(z)\| \leq K_{N}(z) y_{N}(z) dz\|_{\infty}.
\]

But,
Solving this three equations we have polynomials we have:

\[ \begin{align*}
|K_N(z)y(z) - K_N(z)y_N(z)|_\infty & = |K_N(z)y(z) - K_N(z)y_N(z) + K_N(z)y_N(z) - K_N(z)y_N(z)|_\infty \\
& < |K_N(z)||y(z) - y_N(z)|_\infty + |y_N(z)| \\
& - K_N(z)|||y_N(z)|_\infty + |y_N(z)|_\infty \\
\end{align*} \]

Thus, putting this in the last result we get

\[ |y(t) - y_N(t)|_\infty \leq E_f + \sum_{i=0}^j |A_i + E_{k_i}| |y(t) - y_N(t)|_\infty + \rho \sum_{i=0}^j E_{k_i} \]

Hence,

\[ |y(t) - y_N(t)|_\infty \leq E_f + \rho \sum_{i=0}^j E_{k_i} \left( 1 - \sum_{i=0}^j (A_i + E_{k_i}) \right) \]

This complete the proof. □

5. Numerical examples

In this section, some numerical examples are given to illustrate the applicability and accuracy of the proposed method. All the numerical computations have been done using Maple 18.

Example 5.1. Let us first consider the second order pantograph equation [5,6,13]

\[ y''(t) = \frac{3}{4} y(t) + y \left( \frac{t}{2} \right) - t^2 + 2, \quad t \in [0, 1]. \] (14)

subject to

\[ y(0) = 0, \quad y'(0) = 0 \]

The exact solutions of this problem is known to be \( y(t) = t^2 \) [6]. We apply our technique with \( N = 3 \). Approximating (14) with Genocchi polynomials we have:

\[ G(t)(M^2)^2C = \frac{3}{4} G(t)C + G \left( \frac{t}{2} \right) C - t^2 + 2 \] (15)

Also from the initial conditions we have

\[ G(0)C = 0 \quad \text{and} \quad G(0)(M^2)C = 0 \] (16)

Thus, collocating (15) at \( t = \frac{1}{2} \), we get

\[ \frac{57}{8} c_1 - \frac{7}{4} c_1 - \frac{1}{8} c_2 - \frac{23}{16} = 0 \]

From the initial conditions we have \( c_1 - c_2 = 0 \) and \( 2c_2 - 3c_3 = 0 \). Solving this three equations we have

\[ c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{3} \]

Thus, \( y(t) = G(x)C \) is calculated and we have \( x^2 \) which is the exact solution. This result is the same as in [5] We also compare the difference between the exact solution and numerical solutions obtained by [6,13] and present method in Table 1

Example 5.2. We consider the following pantograph equation [10]

\[ y'(t) = \frac{1}{2} t^2 y \left( \frac{t}{2} \right) + \frac{1}{2} y(t), \quad t \in [0, 1]. \] (17)

subject to

\[ y(0) = 0 \]

The exact solution of this problem is known to be \( y(t) = e^t \). We solve (17) using our technique with \( N = 15 \). The solution obtained by our method is in good agreement with the exact solution as shown in Fig. 1. The absolute error obtained by our method is compared with that obtained in [10] as shown in Table 2.

Example 5.3. Consider the following non linear third order pantograph equation

\[ y'''(t) = -1 + 2y' \left( \frac{t}{2} \right), \quad t \in [0, 1]. \] (18)

subject to

\[ y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0 \]

The exact solution of (18) is known to be \( y(t) = \sin(t) \) which we solve using our technique with \( N = 15 \). We compare our result with those obtained using HPM in [6], eight terms of the series have been used which means the terms are up to \( t^{15} \). Hence, in our method we used \( N = 15 \). Fig. 2, also confirm that our solution is in good agreement with the exact solution (see Table 3).

![Fig. 1. Comparison of our solution with the exact solution for Example 5.2.](image-url)

### Table 1

Comparison of approximate solution obtained by [6,13] and present method with exact solution for Example 5.1.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Ref. [13], 13 terms</th>
<th>Ref. [6], 8 terms</th>
<th>Present method, 3 terms (( N = 3 ))</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>0.4</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
</tr>
<tr>
<td>0.6</td>
<td>0.36</td>
<td>0.36</td>
<td>0.36</td>
<td>0.36</td>
</tr>
<tr>
<td>0.8</td>
<td>0.64</td>
<td>0.64</td>
<td>0.64</td>
<td>0.64</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
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</table>

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Example 5.4. Here we consider the delay equation with neutral term [14]
\[ y'(t) = -y(t) + \frac{1}{2} y\left(\frac{t}{2}\right) + \frac{1}{2} y\left(\frac{t}{3}\right) \quad t \in [0, 1]. \]  
subject to
\[ y(0) = 1. \]  
(19)

The exact solution of this problem is given by \( y(t) = e^{-t}. \) We used present method to solve (19) with \( N = 8, 11. \) The absolute error obtained by our method is compared with that obtained in [14] as shown in Table 4.

Note that in Table 4, \( m \) and \( N \) stands for the number of terms of the polynomials used for evaluating the approximate solution of the problem.

Example 5.5. Finally we consider the following neutral delay differential system [9]

Table 2
Comparison of absolute errors obtained by [10] and present method for Example 5.2

<table>
<thead>
<tr>
<th>( t )</th>
<th>Ref. [10] with 15 terms</th>
<th>Ref. [10] with 16 terms</th>
<th>Present method, with ( N = 15 )</th>
<th>Present method with ( N = 16 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>2.220E-16</td>
<td>2.22E-16</td>
<td>3.318E-18</td>
<td>7.785E-20</td>
</tr>
<tr>
<td>0.4</td>
<td>2.220E-16</td>
<td>2.22E-16</td>
<td>4.051E-18</td>
<td>9.506E-20</td>
</tr>
<tr>
<td>0.6</td>
<td>2.220E-16</td>
<td>2.22E-16</td>
<td>4.948E-18</td>
<td>1.161E-19</td>
</tr>
<tr>
<td>0.8</td>
<td>1.332E-15</td>
<td>0.00</td>
<td>6.044E-18</td>
<td>1.418E-19</td>
</tr>
<tr>
<td>1.0</td>
<td>5.018E-14</td>
<td>2.22E-15</td>
<td>7.339E-18</td>
<td>1.741E-19</td>
</tr>
</tbody>
</table>

Table 3
Comparison of the Exact solution and approximate solution obtained by [6] and our method for Example 5.3.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Ref. [6]</th>
<th>Present method</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.19866933079506122</td>
<td>0.19866933079506121678</td>
<td>0.19866933079506122</td>
</tr>
<tr>
<td>0.4</td>
<td>0.38941834230865050</td>
<td>0.38941834230865049281</td>
<td>0.38941834230865050</td>
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<tr>
<td>0.6</td>
<td>0.5646424733950355</td>
<td>0.5646424733950355809</td>
<td>0.56464247339503555</td>
</tr>
<tr>
<td>0.8</td>
<td>0.71735609089952280</td>
<td>0.71735609089952276216</td>
<td>0.71735609089952280</td>
</tr>
<tr>
<td>1.0</td>
<td>0.84147109848078965</td>
<td>0.841471098480789650670</td>
<td>0.84147109848078965</td>
</tr>
</tbody>
</table>

Table 5
Comparison of the Exact solution and approximate solution \( y_1(t) \) obtained by [9] and our method for Example 5.5.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Ref. [9]</th>
<th>Present method</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>1.727110515</td>
<td>1.72711051572</td>
<td>1.727110515</td>
</tr>
<tr>
<td>0.37</td>
<td>1.435646731</td>
<td>1.43564673194</td>
<td>1.43564673194</td>
</tr>
<tr>
<td>0.53</td>
<td>1.657869494</td>
<td>1.65786949482</td>
<td>1.65786949482</td>
</tr>
<tr>
<td>0.79</td>
<td>2.034709939</td>
<td>2.034709939839</td>
<td>2.034709939839</td>
</tr>
<tr>
<td>1.00</td>
<td>2.319768281</td>
<td>2.3197682812471</td>
<td>2.3197682812471</td>
</tr>
</tbody>
</table>

Table 6
Comparison of the Exact solution and approximate solution \( y_2(t) \) obtained by [9] and our method for Example 5.5.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Ref. [9]</th>
<th>Present method</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>1.173510870</td>
<td>1.17351087099</td>
<td>1.17351087099</td>
</tr>
<tr>
<td>0.37</td>
<td>1.447734613</td>
<td>1.44773461466</td>
<td>1.44773461466</td>
</tr>
<tr>
<td>0.53</td>
<td>1.698932307</td>
<td>1.69893230861</td>
<td>1.69893230861</td>
</tr>
<tr>
<td>0.79</td>
<td>2.203396425</td>
<td>2.20339642625</td>
<td>2.20339642625</td>
</tr>
<tr>
<td>1.00</td>
<td>2.718281828</td>
<td>2.71828182846</td>
<td>2.71828182846</td>
</tr>
</tbody>
</table>

Table 4
Comparison of absolute errors obtained by [14] and present method for Example 5.4

<table>
<thead>
<tr>
<th>( t )</th>
<th>Ref. [14], ( m = 8 )</th>
<th>Present method, ( N = 8 )</th>
<th>Present method, ( N = 11 )</th>
<th>Present method, ( N = 15 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>7.08E-4</td>
<td>6.05E-12</td>
<td>6.251E-18</td>
<td>4.133E-18</td>
</tr>
<tr>
<td>0.4</td>
<td>1.29E-3</td>
<td>6.380E-12</td>
<td>6.061E-12</td>
<td>3.990E-18</td>
</tr>
<tr>
<td>0.6</td>
<td>1.76E-3</td>
<td>5.554E-12</td>
<td>5.554E-12</td>
<td>3.629E-18</td>
</tr>
<tr>
<td>0.8</td>
<td>2.15E-3</td>
<td>4.980E-12</td>
<td>4.980E-12</td>
<td>3.237E-18</td>
</tr>
<tr>
<td>1.0</td>
<td>2.47E-3</td>
<td>4.980E-12</td>
<td>4.980E-12</td>
<td>3.237E-18</td>
</tr>
</tbody>
</table>
Table 7
Comparison of the Exact solution and approximate solution $y_4(t)$ obtained by [9] and our method for Example 5.5

<table>
<thead>
<tr>
<th>$t$</th>
<th>Ref. [9]</th>
<th>Present method</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>1.083287066</td>
<td>1.08328706767</td>
<td>1.08328706766</td>
</tr>
<tr>
<td>0.37</td>
<td>1.203218439</td>
<td>1.20321844012</td>
<td>1.20321844012</td>
</tr>
<tr>
<td>0.53</td>
<td>1.303409795</td>
<td>1.3034097577</td>
<td>1.3034097577</td>
</tr>
<tr>
<td>0.79</td>
<td>1.484384190</td>
<td>1.48438419052</td>
<td>1.48438419052</td>
</tr>
<tr>
<td>1.00</td>
<td>1.648721270</td>
<td>1.64872127070</td>
<td>1.64872127070</td>
</tr>
</tbody>
</table>

Table 8
Comparison of the Exact solution and approximate solution $y_3(t)$ obtained by [9] and our method for Example 5.5

<table>
<thead>
<tr>
<th>$t$</th>
<th>Ref. [9]</th>
<th>Present method</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>1.054781178</td>
<td>1.05478118025</td>
<td>1.05478118025</td>
</tr>
<tr>
<td>0.37</td>
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<td>1.13126144525</td>
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<tr>
<td>0.53</td>
<td>1.193232821</td>
<td>1.1932328240</td>
<td>1.1932328240</td>
</tr>
<tr>
<td>0.79</td>
<td>1.301260999</td>
<td>1.30126040013</td>
<td>1.30126040013</td>
</tr>
<tr>
<td>1.00</td>
<td>1.395612425</td>
<td>1.39561242508</td>
<td>1.39561242508</td>
</tr>
</tbody>
</table>

In this paper, a collocation method based on the Genocchi operational matrix for solving generalized pantograph equations is presented. The comparison of the results shows that the present method is an excellent mathematical tool for finding the numerical solutions delay equation. The advantage of the method over others is that it has less computational complexity because every operational matrix of differentiation involves more numbers of zeroes and thus, reduces the run time and provide the solution at high accuracy.

6. Conclusion

In this paper, a collocation method based on the Genocchi operational matrix for solving generalized pantograph equations is presented. The comparison of the results shows that the present method is an excellent mathematical tool for finding the numerical solutions delay equation. The advantage of the method over others is that it has less computational complexity because every operational matrix of differentiation involves more numbers of zeroes and thus, reduces the run time and provide the solution at high accuracy.

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References


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