MATHEMATICAL ANALYSIS OF FRACTIONAL-ORDER CHEMOSTAT MODEL WITH TIME DELAY

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In the name of Allah,

This Master Thesis is dedicated to my beloved mother and father,

Mohd Aris Bin Mohamed Zin and Selbiah Binti Omar and also my siblings

who have always been a constant source of support with prayer, patience and love.

You are the biggest spirit for me.

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ABSTRACT

The fractional-order differential equations (FDEs) is studied to describe the dynamic behaviour of a chemostat system. This study also investigated the FDEs with time delay to examine the effect of time delay on the behaviour of a chemostat system. The integer-order chemostat model in the form of the ordinary differential equation (ODEs) is extended to the FDEs. The stability and bifurcation analyses of the fractional-order chemostat model are discussed by using the Adams-type predictor-corrector method. Furthermore, the behaviour of the fractional-order chemostat system that considered time delay was also observed by using the modified Adams-type predictor-corrector method. The result shows that increasing or decreasing the value of the fractional order, α, may stabilise the unstable state of a chemostat system and also may destabilized the stable state of the chemostat system depend on the predefined parameter values. The increasing the value of the initial substrate concentration, S_0 may destabilise the stable state of a chemostat system and stabilise the unstable state of the system. Therefore, the running state of a fractional-order chemostat system is affected by the value of α and the value of the initial substrate concentration, S_0. In actual application, the value of the initial substrate should remain at $S_0 \geq 2.54$ to ensure that the chemostat system is unstable state. This is because there will be some change in amount of the cell mass concentration whether increase or decrease when the system is unstable, so chemostat system can be well controlled in order to be suitable for cell mass production. Other than that, the convergence speed of nearby trajectories increased when the value of α decreased. These results may be important to fastest the calculation time to achieve the steady-state in order to design the suitable state of the chemostat system. It is also observed that the inclusion of time delay can transform a stable state into a limit cycle and an unstable state with the appropriate choice of time delay value. Therefore, the suitable value of time delay can be chosen properly to ensure the dynamic behaviour of the chemostat system will always be unstable state and hence is suitable for the production of cell mass.

ABSTRAK

Persamaan pembezaan dengan urutan pecahan (PPP) dipelajari untuk menghuraikan perilaku dinamik sistem kemostat. Kajian ini juga menyiasat PPP dengan penangguhan waktu untuk memeriksa kesan penangguhan waktu terhadap perilaku sistem kemostat. Model kemostat dengan urutan nombor bulat dalam bentuk persamaan pembezaan biasa (PPB) diubah kepada PPP. Analisis kestabilan dan analisis bifurkasi untuk model kemostat dengan urutan pecahan telah dibincangkan dengan kaedah peramal-pembetul jenis Adams. Tambahan lagi, perilaku sistem kemostat dengan urutan pecahan yang mempertimbangkan penangguhan waktu diperhatikan dengan menggunakan kaedah peramal-pembetul jenis Adams yang telah diubah. Hasil kajian menunjukkan peningkatan atau penurunan nilai α dapat menstabilkan sistem kemostat yang tidak stabil dan menyahstabilkan sistem kemostat yang stabil. Sementara itu, peningkatan nilai awal kepekatan substrat, S_0 dapat menyahstabilkan sistem kemostat yang stabil bergantung kepada nilai parameter yang ditetapkan. Oleh itu, keadaan sistem kemostat dengan urutan pecahan dipengaruhi oleh nilai α dan nilai awal kepekatan substrat, S_0. Dalam aplikasi sebenar, nilai awal kepekatan substrat mesti kekal pada $S_0 \geq 2.54$ untuk memastikan sistem kemostat tidak stabil. Hal ini kerana, terdapat perubahan jumlah kepekatan jisim sel sama ada meningkat atau menurun apabila sistem tidak stabil, oleh itu, sistem kemostat dapat dikawal dengan baik agar sesuai untuk penghasilan jisim sel. Selain itu, kelajuan penumpuan bagi lintasan berdekatan meningkat apabila nilai α menurun. Keputusan ini penting untuk mempercepatkan masa pengiraan untuk mencapai keadaan seimbang bagi merangka keadaan yang sesuai bagi sistem kemostat. Ia diperhatikan bahawa mempertimbangkan penangguhan waktu boleh mengubah keadaan stabil kepada kitaran had dan keadaan tidak stabil dengan pilihan nilai penangguhan waktu yang sesuai. Oleh itu, nilai penangguhan waktu dapat dipilih dengan baik untuk memastikan perilaku sistem kemostat sentiasa tidak stabil dan sesuai untuk penghasilan jisim sel.

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LIST OF SYMBOLS AND ABBREVIATIONS

F	-	Flow rate
$f(S)$	-	Function response of the prey population on the substrate
$g(X_1)$	-	Function response of the predator population on the prey
Y, Y_1, Y_2	-	Yield coefficient
h	-	Step size
$\Delta X, \Delta S$	-	Impulsive input
j, k, n	-	Iteration
$N(S)$	-	Specific growth rate
t	-	Time
V	-	Reactor volume
υ	-	Configuration of the reactor
$X_1(t)$	-	Prey population concentration
$X_2(t)$	-	Predator population concentration
α	-	Order of fractional derivatives
Γ	-	Gamma
τ	-	Time delay
$P(\lambda)$	-	Characteristic polynomial of eigenvalues
α^*	-	Critical value of fractional-order
$D_{t,b}^{\alpha} f(t)$	-	Right-side Riemann-Liouville and Caputo
$D_{a,t}^{\alpha} f(t)$	-	Left-side Riemann-Liouville and Caputo
$\Re^n$	-	Real number
S_0^*	-	Critical value of parameters
$y_h^P(t_{n+1})$	-	Predictor formula
$a_{j,n+1}$	-	Corrector weight
$b_{j,n+1}$	-	Predictor weight
S_0	-	Initial substrate concentration
Q	-	Dilution rate
X_0	-	Initial cell mass concentration

K_d	-	Death rate of cell mass
t_f	-	Final time
(S_i, X_i)	-	Steady-state solution
$y_h(t_{n+1})$	-	Corrector formula
$\lambda, \lambda_1, \lambda_2$	-	Eigenvalues
X , $X(t)$	-	Cell mass concentration
S , $S(t)$	-	Substrate concentration
μ, μ_1, μ_2	-	Maximum specific growth rate
K, K_1, K_2	-	Saturated constant
$\beta, \theta, \gamma, \phi$	-	Constant in yield coefficient
ρ	-	Constant of steady-state solution
b , c	-	Constant of characteristic polynomial
p , q	-	Constant of complex conjugate roots
P_1 , P_2	-	Constant of stability condition
ODEs	-	Ordinary Differential Equations
FDEs	-	Fractional-order Differential Equations
DDEs	-	Delay Differential Equations
IS-LM	-	Investment and Saving-Liquidity Preference and Money Supply

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Background of research

Biological continuous culture is a part of the topics in microbiology as well as in mathematical biology. Many mathematical models have been developed to predict and study the biological system. In the past four decades, there have been far-reaching research on improving cell mass production in chemical reactors (Nelson & Sidhu, 2005). Among those models, the chemostat model is one of the models used to understand the mechanism of cell mass growth in a chemostat. Moreover, the production of cell mass has a high demand due to its environmental properties and its advantages. Hence, the mechanism of cell mass growth needs to keep being improved in order to fulfil the demand.

A chemostat is an apparatus for continuous culture that contains bacterial populations. A chemostat can be used to investigate cell mass production under controlled conditions. This reactor provides a dynamic system for population studies and is suitable to be used in a laboratory. The diagram of a chemostat can be depicted as shown in Figure 1.1. A substrate is continuously added into the reactor containing the cell mass, which grows by consuming the substrate that enters through the inflow chamber. Meanwhile, the mixture of cell mass and substrate is continuously harvested from the reactor through the outflow chamber. The dynamics in the chemostat can be investigated by using the chemostat model (Harmand *et. al*, 2017).

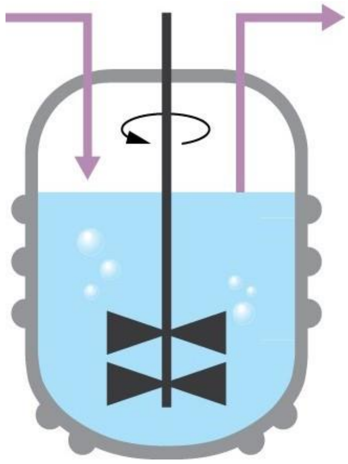

Figure 1.1: Chemostat

In the chemostat model, the relationship between the specific growth rate of cell mass and substrate is explained by using the microbial kinetics model. The microbial growth kinetics model describes the behaviour of the cell mass growth process in terms of mathematical equations (Sakthiselvan, Meenambiga & Madhumathi, 2019). There are many types of microbial growth kinetics models, such as the Monod, Tessier, Moser, Contois and Andrews models (Monod, 1949; Moser, 1958; Contois, 1959; Andrews, 1968). The yield coefficient is a function of the substrate concentration, which can express the cell mass formation. The yield coefficient can be demonstrated as the cell mass formed per unit mass of the substrate consumed (Hong, 1989). Some studies also considered the variable yield coefficient in investigating the cell mass growth in a chemostat (Lee, Fredrickson & Tsuchiya, 1976; Dorofeev *et al*., 1992). A model with variable yield coefficient is suitable for the growth of a continuous culture (Suzuki, Shimizu & Matsubara, 1985).

Ordinary differential equations (ODEs) are commonly used for modelling biological systems. However, the behaviour of most biological systems has memory effects which means how much information the system carries, and ODEs usually neglect such effects (Zeinadini & Namjoo, 2017a; Zeinadini & Namjoo, 2017b). Therefore, fractional-order differential equations (FDEs) are taken into account when describing the behaviour of the systems' equations. Many biological systems highlighted that the state of a physical system not depends only on its current state but also depends on its historical states. A FDEs is a generalisation of the ODEs to random nonlinear order (Elettreby, Al-Raezah & Nabil, 2017). This equation is more effective because of its good memory, among other advantages (Cui *et al*., 2016; Zeinadini *et. al.*, 2017a; Zeinadini *et. al.*, 2017b; Garrappa, 2018; Islam *et al*., 2020). In addition, the errors arising from the disregarded parameters when modelling real-life phenomena also can be reduced. FDEs are also used to efficiently replicate the real nature of various systems in the field of engineering and sciences (Nelson & Sidhu, 2009). In the past few decades, FDEs have been used in biological systems (Fu, Ma & Ruan, 2005; Ahmed, Hashish & Rihan, 2012; Ferdi, 2012; Eslami *et al*., 2014; Sidhu, Nelson & Balakrishnan, 2015; Alqahtani, Nelson & Worthy, 2015; Zeinadini & Namjoo, 2017a; Zeinadini & Namjoo, 2017b; Ezz-Eldien, 2018; D'Ovidio, Loreti & Ahrabi, 2018; Khater, Attia & Lu, 2019; El-Hajji & Sayari, 2019), physics (Yang *et al*., 2019; Sun *et al*., 2018), medicine (Guan *et al*., 2018; Sopasakis *et al*., 2018), finance (Huang, Cai & Cao, 2018), hydrology (Tu, Ercan & Kavvas, 2018) and chemistry (Yuste, Acedo & Lindenberg, 2004; Arafa, Rida & Khalil, 2012; Sarwar & Iqbal, 2018). There are many recent developments on FDEs with the application of various operators (Oliveira & Machado, 2014). Since great strides in the study of FDEs have been developed, the dynamics in the chemostat can be investigated using the mathematical model of the chemostat in the form of FDEs.

The study of the FDEs is receiving increasing attention in recent years. However, the effective dimension in FDEs is one of the important problems that concern in this field (Bhalekar & Daftardar-Gejji, 2010). A delay is able to express the information from the earlier state and provide a history of the system over the delay interval $[-\tau, 0]$ as initial condition (Bhalekar & Daftardar-Gejji, 2010; Jhinga & Daftardar-Gejji, 2019). These cause delay systems are infinite dimensions in nature. Sometimes, internal and external uncertainties in the application, such as time delay, cannot be avoided and may lead to instability (Cui *et al*., 2016). Hence, it can be of foremost significance to consider time delay. Delay can be recognised everywhere and is often encountered in many practical systems such as biology, economics and automatic control (Wang, 2013). Because of its wide usefulness, this study aims to deepen the analysis of the integer-order chemostat model with fractional-order theory and to also consider time delay.

Therefore, in this study, the stability of the equilibrium points of the fractional-order chemostat model is discussed. Then, the bifurcation analysis for the fractional-order chemostat model was conducted to identify the bifurcation point that can change the stability of the system. The analysis identified the values of the fractional-order and the system parameters to ensure the operation of the chemostat is well control. Numerical simulation of the fractional-order chemostat model with time delay was also conducted to observe the effect of time delay on the behaviour of the fractional-order systems.

1.2 Problem statement

A chemostat, or a continuously stirred bioreactor, is a tool used for the continuous production of cell mass. A chemostat model is a coupled set of differential equations derived by using the mass balance law.

Integer-order differential equations have been mostly used to represent the mathematical model of a biological system, such as the chemostat model. However, a mathematical model based on integer-order differential equations does not accurately model the complete system behaviour due to the fundamental properties of the system (Choudhuri, 2018). Moreover, the behaviour of most biological systems has memory effects, and integer-order differential equations usually neglect such effects. Meanwhile, FDEs have more benefits for the explanation of memory and hereditary properties of a system (Cui *et al.*, 2016). Therefore, FDEs have been studied to examine the evolution trend and dynamic behaviour of a chemostat system.

FDEs tend to lower the dimensionality of a system (Daftardar-Gejji, Bhalekar & Gade, 2012). However, the dimensionality can be infinite-dimensional if time delay is considered in the differential equation. This shows that time delay affects the dimension and the running state of a fractional-order chemostat system. Yet, there have been few studies on the expansion of the chemostat model with fractional-order theory as well as fractional-order time delay. Therefore, the FDEs with time delay is studied to investigate the effect of time delay on the dynamic behaviour of a fractional-order chemostat system.

1.3 Research objectives

The objectives of this research are:

i. to analyse the dynamic behaviour of the fractional-order chemostat system, and
ii. to analyse the dynamic behaviour of the fractional-order chemostat system with time delay.

1.4 Scopes of research

Many mathematical models have been developed to study the chemostat model. Nelson and Sidhu (2005) studied the integer-order chemostat model of single cell growth with variable yield coefficient and considering Monod growth model. Meanwhile, Zeinadini & Namjoo (2017a) and Zeinadini & Namjoo (2017b) studied the fractional-order chemostat model of two cell growth with variable yield coefficient and considering Monod growth model. This research is referring to these three papers frequently. In this research, the chemostat model for single cells growing through the consumption of a substrate was extended to the fractional-order model. No initial amount of cell mass was assumed to be supplied into the chemostat $\left(X_0 = 0\right)$. Here, the Caputo derivatives definition of order $0<\alpha<1$ was adopted. The Monod growth model was considered in this research to determine the growth rate of the microbial. The yield coefficient was defined as a function of the substrate concentration instead of a constant. The stability analysis of the chemostat model was determined with the fractional-order derivative. Based on the equilibrium point of the chemostat model, the eigenvalues were classified as either stable or unstable according to their parameter condition by referring to the respective theorem and proposition. Then, the bifurcation analysis was also presented with the fractional-order derivative by considering only the Hopf-type bifurcation. Hopf-type bifurcation is a bifurcation point where the stability of the system switches and a periodic solution arises. Due to this reason, Hopf-type bifurcation is the most related to this research finding. Here, the fractional order, α, and the initial concentration of the substrate, S_0, were chosen as bifurcation parameters. The theorems of Hopf-type bifurcation is frequently referred from Li and Wu (2014) and Ma and Ren (2016). All analyses of stability and bifurcation were simulated using the Adams-type predictor-corrector method. In addition, this research also studied the fractional-order chemostat model with time delay by numerical simulation only. Simulation of the fractional-order chemostat model with time delay was simulated using the modified Adams-type predictor-corrector method. The results are presented graphically using the Mathematica and Maple software.

1.5 Significance of research

In this research, the FDEs was implemented to deepen the analysis of the chemostat system. The fractional-order has significant effects on the running state of the chemostat system. Analyses of the dynamic behaviour of the fractional-order system included stability and bifurcation analyses. Stability analysis was needed to determine the existence of steady-state solutions in the chemostat system. Meanwhile, bifurcation analysis would present the critical point, or the bifurcation point, where the behaviour of the chemostat system changed. Thus, bifurcation analysis would provide the suitable values of the fractional order and the parameters so as ensure the controllability and stability of the chemostat. The change in the parameters' value would show the change in the chemostat system behaviour. Therefore, the operation of the chemostat would be well controlled in order to be suitable for cell mass production.

Time delay was considered to determine the effect of time delay on the behaviour of the fractional-order chemostat system. The simulation of the dynamic behaviour of the fractional-order system with time delay included the numerical analysis. The destabilisation of a stable state by suitable time delay values was verified. Thus, appropriate values of time delay could be obtained to ensure the operation of the chemostat system is well controlled.

1.6 Framework of research

This thesis consists of six chapters. The first chapter discusses the background of research, the problem statement, the research objectives, the scope of research, the significance of research and the framework of research.

In Chapter 2, the literature review of this research is presented. The first topic in this chapter is the FDEs, where its definition, advantages and development are discussed. This chapter also provides a discussion on the history of FDEs, including the Riemann-Liouville and Caputo definitions. Next, the mathematical models of the chemostat system and the fractional-order chemostat system are also discussed. Finally, the mathematical model of the chemostat system with time delay and the mathematical model of the fractional-order system with time delay are discussed in this chapter.

In Chapter 3, the methodology of this research is provided. The stability analysis of the fractional-order system is studied, where its theorem and proposition are described. In addition, the Hopf bifurcation analysis of the fractional-order system is also discussed, including the Hopf bifurcation analysis of the fractional order and the parameter values. Then, numerical schemes for solving the fractional-order chemostat system as well as the fractional-order chemostat system with time delay are discussed in detail.

Next, Chapter 4 discusses the stability and bifurcation analyses of the fractional-order chemostat system. The stability analysis of the fractional-order chemostat model was studied based on the stability theory of FDEs system. The conditions in the stability theory of FDEs system needed to be satisfied in order to get a stable steady-state solution. Then, the Hopf bifurcation analysis was conducted to identify the bifurcation point that can change the stability of the system. All simulation of stability and bifurcation were simulated using the Adams-type predictor-corrector method.

Chapter 5 presents the numerical simulation of the fractional-order chemostat model with time delay. The behaviour of the fractional-order chemostat system with time delay was observed. The simulation for the numerical analysis of the fractional-order chemostat model with time delay was investigated by using the modified Adams-type predictor-corrector method. The result was estimated and visualised by using the Maple software.

Lastly, Chapter 6 consists of the summary of this research and some recommendations for future work. The framework of this research can be depicted in Figure 1.2.

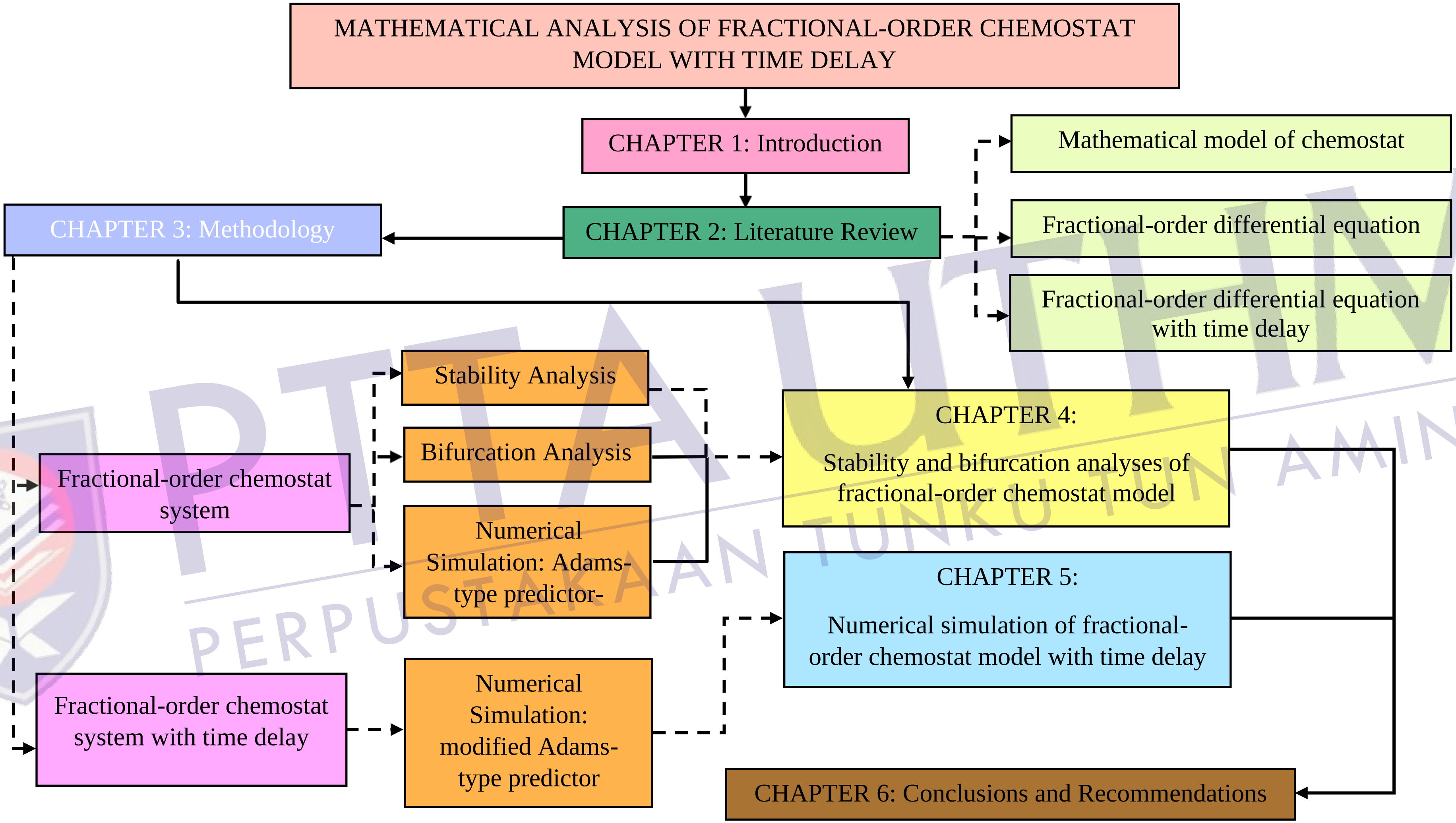

Figure 1.2: Framework of research

CHAPTER 2

LITERATURE REVIEW & BASIC CONCEPT

2.1 Fractional-order differential equation

Fractional-order differential equations (FDEs) can be defined as the rationalisation of classical differential equations involving the differentiation and integration of non-linear order (Lazarevic *et al.*, 2014). The history of FDEs started in 1695 when mathematician Guillaume de L'Hôpital sent a letter to his fellow mathematician Gottfried Wilhelm Leibniz regarding the answer to function derivatives if the order of the derivative was a non-integer (Ross, 1977), which Leibniz was then able to predict. Subsequently, FDEs attracted the interest of other well-known mathematicians, such as Liouville, Lagrange, Riemann, Euler, Grünwald, Letnikov, Caputo, Holmgren, Heaviside, Abel, Fourier and many others. The theory of FDEs has developed rapidly since the 19th century. Nowadays, the applications of FDEs are far-reaching. The application of FDEs are becoming more relevant in various fields of science, including engineering, numerical analysis, physics, economics and biology (Ögrekçi, 2015). Almost no course in modern science and engineering, in general, remain untouched by the techniques and the tools of FDEs (Lazarevic *et al.*, 2014).

It is known that FDEs provide a more powerful tool for explaining and modelling many complex systems in society and nature as compared with ordinary differential equations (ODEs). ODEs are unable to characterise most complex systems' process and behaviour. Many complex systems' mathematical modelling involves the long-memory in time and non-local dynamics. FDEs have some of these characteristics. Therefore, FDEs are probably the most useful tool from the application and the mathematical standpoints (Bonilla, Rivero & Trujillo, 2007). In other words, FDEs translate the reality of nature better. FDEs are also able to reduce

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