SOLVING TRANSPORT-DENSITY EQUATION WITH DIFFUSION USING THE FIRST INTEGRAL METHOD AND THE GENERALIZED HYPERBOLIC FUNCTIONS METHOD

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I hereby declare that the work in this thesis is my own except for quotations and summaries which have been duly acknowledged

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Learning never exhausts the mind…….Leonardo da Vinc

Stand up to your obstacles and do something about them. You will find that they haven't half the strength you think they have.

My humble effort I dedicate to,

My lovely mother and father,

My mother and father were both much more remarkable than any story of mine. They seem to me just gifts from God.

My lovely wife

I would like to tell the world how grateful I am for all the love and sacrifices my wife has made.

My daughters

Who taught me to be more patient and always give me the hope in life.

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ABSTRACT

This study gives an overview of nonlinear partial differential transport-density equation with diffusion traffic flow model. Basically, the first integral method (FIM) and the generalized hyperbolic functions method (GHFM) are employed to solve the proposed model and to give compelling evidence that regularization of the conservation law by adding viscosity that will undeniably remove the singularity and the weak solution obtained by the characteristic method. As long as, continuity equation leads to discontinuous solutions, abrupt change of the traffic density; thus the diffusion was introduced so as to prevent incrementally deformation of the wave so that shock, rarefication and singularity will not be existed anymore. Subsequently, smooth the resulting density field between the two asymptotic states. On the base of the solution, physical interpretations for some obtained solutions were discussed in order to detect the effects of diffusion on this dynamical traffic flow model.

ABSTRAK

Kajian ini memberikan gambaran tentang persamaan ketumpatan-pengangkutan pembezaan separa tak linear dengan model aliran trafik penyebaran. Pada dasarnya, kaedah kamiran pertama (FIM) dan kaedah fungsi hiperbolik umum (GHFM) digunakan untuk menyelesaikan model yang dicadangkan dan memberikan bukti yang kuat bahawa pengaturcaraan hukum pemuliharaan dengan menambah kelikatan akan tidak dinafikan menghapuskan singulariti dan penyelesaian lemah yang diperolehi oleh kaedah cirian. Selagi persamaan keselanjaran membawa kepada penyelesaian tak selanjar, perubahan mendadak bagi ketumpatan trafik; maka penyebaran diperkenalkan bagi mengelakkan perubahan bentuk gelombang secara berperingkat supaya kejutan, rarefikasi dan singulariti tidak akan wujud lagi. Selanjutnya, melicinkan medan ketumpatan yang terhasil di antara dua keadaan asimtotik. Pada asas penyelesaian, tafsiran fizikal untuk beberapa penyelesaian yang diperoleh dibincangkan untuk mengesan kesan penyebaran pada model aliran trafik dinamik ini.

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LIST OF SYMBOLS AND ABBREVIATIONS

c_1, c_2, k	-	Non-zero Constants
d	-	The viscosity
e		Neperian number
f	-	Wave function
M		Number of cars
Q	-	Traffic flow
t	-	Time
v	-	Velocity
v_m	-	Maximum velocity
x	-	The spatial coordinate

Greek symbols

ε	-	Characteristic variable
ω, λ, μ	-	Non-zero Constants
ρ	-	Fluid density
ρ_m	-	Maximum density
ρ_l	-	Density on the left
ρ_r	-	Density on the right
σ	-	Wave speed
$\psi(x)$	-	Function of the variable x

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CHAPTER 1

INTRODUCTION

1.1 Research background

Traffic jams have become commonly faced problem over the last few decades. Different aspects of the mathematical approaches of vehicular traffic have raised the interest of many researchers such as in the books by Whitham (1974), Mesterton-Gibbons (1989), Fowkes and Mahony (1994), Haberman (1998) and Knobel (2000). Therefore, using either mathematical or simulation models, several issues related to traffic flow could be tackled. For example, the traffic after the traffic light turned red or green, crossroads, avoiding a collision with the last car in a queue, T-junctions, all round traffic etc. In mathematical modeling, traffic flow can be modeled by a nonlinear partial differential equation (NLPDE) called viscous Burgers' equation in the name of Johannes Martinus Burgers (Burgers, 1948). So far, viscous Burgers' equation is considered as one of the most important PDEs, since it is a combination between nonlinear wave equation and linear diffusion, moreover it reflects notable example of physical phenomena.

Many scientists have persistently sought to find the best solution for Burgers' equation in the field of traffic flow. Beskos *et al*. (1985) used finite element method (FEM) to analyze traffic flow. By comparison, between the FEM and the finite difference method (FDM), the authors found it slightly more accurate by using FEM than FDM. However, they admitted that FEM was more time-wasting than first and second order FDM. Helbing and Treiber (1999) used the numerical integration, concluding that the explicit methods are less powerful but computationally easier.

Jin and Zhang (2001) studied the Payne-Whitham (PW) model in terms of relaxation time. In particular, the Riemann problem so the authors outlined that the solutions for LWR model (Lighthill and Whitham, 1955, Richards, 1956) and PW model were almost equivalent only if the PW model was stable. Additionally, they illustrated when the relaxation time approaches zero, the PW model reduces to the viscous LWR model. Meanwhile, Sakai and Kimura (2005) studied the advection–diffusion transport equations numerically and they used Cole–Hopf transformation to linearize the advection term so that the authors obtained a solution to the initial value problems, Moreover, they indicated when the diffusivity coefficient $d \to 0$ the solution will be discontinuous as result of vanishing viscosity method. Providentially, some scientists like Jia *et al*. (2012) solved most of these problems by adding artificial viscosity because in order to reflect the fact that drivers will decrease their speed when the density increases ahead, one must assume that the flow Q is a function of the density gradient, they claimed.

Nonlinear analysis clearly shows that the density fluctuation in traffic flow leads to a variety of density waves so the Korteweg–de Vries–Burgers (KdV–Burgers) equation was derived to describe the traffic flow near the neutral stability line (Ge *et al*., 2013). Recently, traffic entropy theory was commonly used to evaluate the transportation systems by constructing a statistical entropy model based on the proportion of the distance between the cars, concluding that a large proportion of the heavy goods vehicles corresponded to a large traffic flow entropy (Liu *et al*., 2019).

1.2 Problem statement

In recent decades, significant advances have been achieved in the domain of NLPDEs. Since then, many critical problems have been posed in the field of physics, fluid dynamics, biology, engineering, geochemistry, turbulent fluid and finance. How much time, fuel and money are wasted on traffic? Traffic congestion has been increasing all over the world in recent years, consequently, more fuel consumption, monetary losses and environmental pollution as well. When a traffic jam exists we are required to be well educated of the traffic factors, especially the density on the roads.

One of the best NLPDEs depicts these phenomena is the transport-density equation which was considered a special case of viscous Burgers' equation. Viscous Burgers' equation exists in other areas, such as in the theory of shock waves, gas dynamics, fiber optics and fluid dynamics. As a non-productive activity for most residents, congestion reduces regional economic health. Furthermore, it decreases our ability to forecast travel time accurately, which causes drivers to allocate more time to travel. As such, transport-density equation plays an immense role in addressing large numbers of traffic engineers' questions. Building such an active traffic continuum model requires a deep knowledge of solving NLPDEs related to transport equations and various traffic flow parameters. Hence, familiarizing with the factors that influence traffic organization and route optimization, such the viscosity coefficient, is extremely necessary.

Owing to discontinuity of the continuity equation as shown in Figure 1.1, viscosity was urgently added to transport-density equation for the sake of eliminating discontinuities of density function mathematically, and physically to reflect the assumption that each driver reacts to a stimulus from other vehicles in a particular different way, based on the correlations between the successive automobiles.

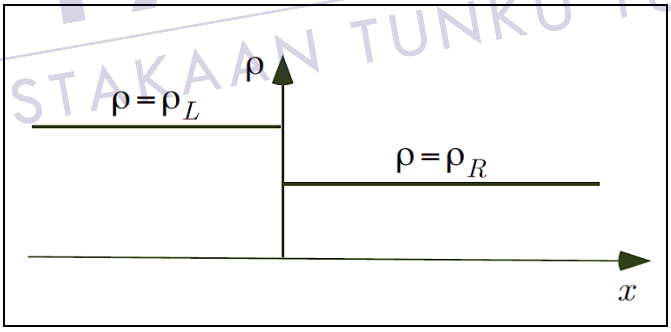

Figure 1.1: Initial density by using Riemann problem

To sum up, this research was about solving 1D nonlinear partial differential transport-density equation with diffusion, and it was developed by describing the evolution of the macroscopic quantities such as traffic density ρ, traffic speed v and traffic flow Q, in order to obtain exact travelling wave solution by using first integral method (FIM) and generalized hyperbolic functions method (GHFM). As long as, the continuity equation leads to discontinuous solutions and allows shocks to be formed, the diffusion term has thus been added to prevent shock formation and the singularity.

1.3 Research questions

This study was basically built to boost car-following model. However, many questions have arisen about this model. For example, how can someone solve the transport-density equation by using the continuity equation, what will happen if we add the viscosity coefficient to the continuity equation. Moreover, what is the physical meaning if singularity in the density field was obtained? Will diffusion term address the discontinuity and shock problems.

On the other hand, if the viscosity coefficient was the key to handle these entire problems, could the resulting NLPDE be solvable. And which method will be efficient to construct traveling wave solutions. Yet, are the first integral method and the generalized hyperbolic functions method considered as powerful tools to obtain exact solution for the nonlinear transport-density equation.

1.4 Research objectives

The objectives of this research are:

1. To apply the transport-density equation.
2. To show the singularity, shock or rarefication with no diffusion model.
3. To find solution of transport-density equation traffic flow type by using the first integral method (FIM).
4. To find solution of transport-density equation traffic flow type by using the generalized hyperbolic functions method (GHFM).
5. To analyze the influences of viscosity on the traffic behavior.

1.5 Scope of study

Mainly, this research concentrated on solving nonlinear partial differential transport-density equation with diffusion traffic flow type which is a prototype for conservation equations. However, the conservation equation without diffusion was solved by the characteristic method to elucidate the singularity, and then by adding viscosity coefficient and using the first integral method and the generalized hyperbolic functions method, we confidently demonstrate the viscosity effect on smoothing the density profile.

Throughout this study, $\rho(x,t)$ was used to refer the density of cars on a highway single lane and the traffic flow by $Q(x,t)$, this model was designed as continuum of mechanics so the conservation equation which merely states the amount of contaminant flows through a section of road changes in time by a flux. Greenshields *et al*. (1935) presented a constitutive law of velocity $v = v_m - \frac{v_m}{\rho_m} \cdot \rho$ which played a pivotal role in this work; moreover, the hydrodynamic flow relation $Q = \rho \times v$ where the velocity is a function of traffic density $v(\rho)$ was used too. Simulation of the obtained solutions of traffic flow problems was provided and different modeling perspectives were also added.

A mathematical macroscopic traffic flow model known as LWR was used in order to produce a quasilinear first order partial differential equation. Additionally, investigation of the effects of the viscous term by numerical simulations was also carried out at the end in order to verify that the viscosity may induce oscillations but prevent unrealistically sharp transitions, specifically shocks.

The empirical results reported herein should be considered in the light of some limitations. For example, vehicles were taken in this model with similar dimensions so motorcycles and heavy trucks were not included, moreover, systematic lane changes were not allowed in this model. Furthermore, the impact of meteorological conditions on the traffic flow such as heavy rain was not addressed in the proposed model. Another downside regarding our model is that we have not systematically examined the behavior of this model for all types of roads.

1.6 Significance of study

In this study, the FIM and GHFM were applied to solve transport-density equation, $\rho_t + \left(v_m - \frac{2v_m}{\rho_m} \rho \right) \rho_x - d \cdot \rho_{xx} = 0$ which is significant in engineering especially fluid mechanics field. Similarly, this research has provided a decisive advantage to other researchers in this area, in particular, traffic engineers so that they can make the right choice in constructing new roads. Consequently, minimization of traffic congestion, time-wasting, air pollution, noise pollution, and maximization of the traffic flow. The advantage of using FIM and GHFM for solving nonlinear partial differential equation is that, both methods could be extended to solve some nonlinear evolution equations related to mathematical physics.

In summary, the major concern of this research is to improve a mathematical model of governing equation to be ideal for describing flow with diffusion.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The usage of partial differential equations happens normally in many applications; they model the rate of change of a physical quantity with respect to several variables. PDEs are usually used to build models of the most fundamental theories in physics and science. Almost all phenomena in physical science can be modeled by mathematical equations. In many cases, this modeling drives us to partial differential equations. Moreover, PDEs play a major role in understanding physical science difficulties allied to flow, fluid mechanics, heat, electromagnetic, theory of shock waves, gas dynamics, fiber optics, fluid dynamics, chemical engineering , elasticity and so on.

Partial differential equations describe a relation between an unknown function and its partial derivatives. They exist normally in all areas of physics and life science. Not long ago, we have seen numerous growths for the use of PDEs in areas such as biology, space, finance and computer sciences describing the interaction between independent variables. According to Pinchover & Rubinstein (2005) when the value of the unknown function(s) at a certain point depends only on what happens in the vicinity of this point, we shall, in general, obtain a PDE. Historically, the study of partial differential equations started in the 18th century in the work of Euler, d'Alembert, Laplace and Lagrange, as the principal approach of analytical study of models in the physical science. Since the middle of the 19th century, PDEs have become, particularly with the work of Riemann, a fundamental tool in many branches of mathematical studies.

Furthermore, PDEs model complex behaviors in the social sciences, for instance conservation laws for traffic flow, population dynamics, stochastic models hydrodynamics and economy. Building a suitable model of PDEs requires deep awareness of mathematics, and supportive evidence from experimental data. However, PDEs have been existed almost inasmuch as derivatives. Soon after people inaugurated computing derivatives, they realized how exceptionally beneficial it would be to overturn the process. For example, the heat equation, the wave equation, Schrödinger's equation, Burgers' equation etc. Clearly, the antiderivative technique was enormously helpful for the traffic flow researchers.

For a long time, LWR model was considered one of the best description for the flux, with the passage of time it seemed that it was not sufficient enough to analysis the data collected from the highways due to too many flawed and unrealistic assumptions, for example, the car velocity adapts immediately to the desired velocity, the shock wave is not realistic in real situation, discontinuity for some functions, limitless acceleration, dispersion and so on. The fact that shocks still exist in the inviscid models has led many scientists to create an even more sophisticated description for the traffic models (Zhang, 2003). Therefore, a small viscosity term was added to the PW model to obtain the viscous model of Kühne (1984). The viscous model choice has become an interested area for many physicists. As such, many publications have been produced about this model like Kerner and Konhäuser (1993). As reported by Pinchover and Rubinstein (2005) the inviscid model induced solutions with a special singularity called a shock wave. Remarkably, the most significant concept in all researches was adding viscosity term.

One of the most fundamental PDE is viscous Burgers' equation or Bateman–Burgers' equation which occurs in different areas of applied mathematics, such as traffic flow. The equation was first introduced by Harry Bateman in 1915 and then studied by Johannes Martinus Burgers (1948). It was considered the simplest model for analyzing merged effect of nonlinear advection and diffusion. However, the first traffic flow model was LWR and has been extensively used for several traffic roads. LWR model itself was based on the conservation of cars and described by the continuity equation.

2.2 Related works

Jia *et al*. (2012) studied the influence of viscosity upon the transport-density equation and traffic behavior, depending on the LWR model, they started with the scalar conservation law to conclude that, the problem of speed adaption is clear, which means the drivers have no time to change their velocity, consequently they suggested taking speed variation. However, the authors relied on diffusive corrected kinematic model into LWR model in order to find the travelling wave solution by assuming the traffic flux is a function of both density and its gradient which leads to speed distributions $v_1, v_2, \ldots$. Hence, the authors claimed that this model represents more plasticity toward drivers' velocity which leads to say that the drivers have the ability to travel slower or faster in comparison with the average speed.

Azam *et al*. (2014) tried to approach the second order traffic flow model with diffusion numerically by using boundary condition and initial condition. Starting with continuum macroscopic model, the authors then updated their model to include the term $d\nabla\rho$ in order to represent nearly all the factors those could play an important role in increasing or decreasing the drivers' velocity. For example, road condition, driver's behavior toward density ahead, cars condition, etc. However, the obtained solution was included a complex double integration so they recommended the finite different method using initial and boundary condition. Finally, they succeeded to represent numerical discretization to demonstrate the propagation effect for different diffusion coefficients, on density profile.

Rhebergen (2005) employed the discontinuous Galerkin finite element method with the intention of finding a numerical solution for the viscous Burgers' equation. Assuming the travelling wave solution is $\varepsilon = x - st$, the author focused primarily on the periodic boundary condition and the initial condition. Then by using Gauss method, Newton method and centered difference scheme the author found approximately the same solution for the given parameters.

Ali *et al*. (2015) had different approach to the macroscopic traffic flow model in a single lane by including a source term. By using LWR model they defined a hyperbolic conservation law including the source term $m(t, x, \rho)$ which represents influx or outflow on a fixed point of the road. In order to make things easier the authors assumed that influx or outflow on a fixed point of the road is constant. Yet, they stated it was entirely important to find the density at a specific time in order to solve the previous equation, therefore, they used the characteristics technique and the Cauchy problem. Nevertheless, the authors were not convinced enough by this solution, because in real life, it seemed difficult to determine the initial density as a function of time.

On another note, Takači (2005) had different perspective where he used the logarithmic model to represent traffic flow in the Lincoln tunnel in New York, so that the author reduced the vehicular collisions associated with traffic jams and improved policies related to traffic laws. The author used the geometric-calculus method to approach the traffic flux with diffusion and applied the hydrodynamic flow relation $Q = \rho.v$ on the conservation law in order to include the number of vehicles travel in/out within an interval of time and on a section of road. So far, the author argued the previous model due to the instant changes of cars velocity so he suggested modifying the model to contain more coefficients. Finally, Takači solved his model with the help of G-calculus, specifically by G- Laplace transformation.

Holmes (2009) had stretched his research about the traffic theory, going through directed motion not only for the cars on the highway but also the blood cells in arteries and other one-dimensional route. Due to many relations between velocity and density Holmes tried hard to find the best approach and postulate how the velocity v and the density ρ are related based on experimental evidence. Therefore, he used a Newell constitutive law instead of Greenshields law to find better relation between density and velocity.

Hartono *et al*. (2018) were fascinated to figure out the density-flux relationship in macroscopic model because it is more realistic and it reflects the actual situation on the highway. Basically, they completed their research depending on Greenshields' model in the framework of first order partial differential equations, by using minor disturbance of velocity which leads to change the quantity of density; as a result, a critical problem raised known as perturbation problem. By using multiple-scale in a different manner, the authors built their model considering one road without any exit or entrance.

2.3 The first integral method (FIM)

In recent times, the first integral method (FIM) has been widely applied to solve various PDEs like, Klein-Gordon-Zakharov equation by Zhang (2013), conformable fractional differential equations by Ilie *et al*. (2018) and higher order nonlinear Schrödinger equations (Zhang *et al*., 2019). Basically, Feng (2002) was the first one who formulated and introduced the first integral method which was built based on the theories from commutative algebra. Starting with the wave transformation with the aim of converting the proposed PDE into a system of ODE, then by using the Division Theorem one can obtain the first integral to the ODE system.

2.4 The generalized hyperbolic functions method (GHFM)

In the recent decade, several methods for finding the exact solutions to NPDEs have been proposed. Many mathematicians made efforts to seek more exact solutions to solve the nonlinear PDEs, perhaps the generalized hyperbolic functions method (GHFM) is one of the powerful methods which was first proposed by Gao & Tian (2001) by using hyperbolic fibonacci and lucas functions, since then the GHFM was used by many researchers such as Al-Muhiameed and Abdel-Salam (2012), Wazzan (2015), Malinzi and Quaye (2018), Hosseini et al. (2018).

In the end, this work was developed using the first integral method and the generalized hyperbolic functions method in order to find a better approach for the traffic flow model with viscosity.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

Under substantial future demands for estimating the traffic flow behavior in real world, traffic capacity needs to be analyzed for specific peak hours. With the purpose of doing that, traffic macroscopic model will be studied in order to solve the nonlinear partial differential equation with diffusion in traffic flow by using the first integral method (FIM) and the generalized hyperbolic functions methods (GHFM). Therefore, the procedures are as follow,

(i) Applying the transport-density equation with singularity, shock or rarefication and introduce the weak solution.
(ii) Using the first integral method in order to solve the transport-density equation.
(iii) Using generalized hyperbolic functions method in order to solve the transport-density equation.
(iv) Taking a numerical simulation and make some comparisons between the previous methods.
(v) Conclusion to summarize the effects of diffusivity on the proposed wave model.

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