ALGEBRAIC STUDY OF FUZZY FINITE SWITCHBOARD AUTOMATA

NUR AIN BINTI EBAS

A thesis submitted in
fulfilment of the requirement for the award of the
Doctor of Philosophy in Science

Faculty of Applied Sciences and Technology
Universiti Tun Hussein Onn Malaysia

NOVEMBER 2020

DEDICATION

I dedicated this thesis,
To my beloved mother and father,
Ebas Bin Hj. Jusoh
Salmiah Binti Chacini
Thank you for all the sacrifices and love that you devoted.

To my beloved husband,
Muhamad Shahril Bin Mohd Abdullah
Who has been considerate, cooperative and supportive throughout the duration of my study.

To my beloved supervisors,
Associate Prof. Dr. Nor Shamsidah binti Amir Hamzah, Associate Prof. Dr. Kavikumar s/o Jacob and Associate Prof. Dr. Mohd Saifullah bin Rusiman for nonstop giving and sharing of knowledge.
Friends and colleagues,
Thank you for all the encouragement, support and guidance.

ACKNOWLEDGEMENT

First and foremost, all praise to Allah SWT the Almighty, for giving me the blessings throughout my research work and also give me strength to complete the research successfully.

I would like to express my deep and sincere gratitude to my research supervisor, Associate Professor Dr. Nor Shamsidah Binti Amir Hamzah, Associate Professor Dr. Kavikumar S/O Jacob and Associate Professor Dr. Mohd Saifullah Bin Rusiman, for giving me the opportunity to do research and providing invaluable guidance throughout this research. They taught me the methodology to carry out the research and to present the research works as clearly as possible. It was a great honor to work and study under them. Next gratitude also goes to all the panels for their constructive comments and inputs as well as encouragement.

I would like to express my special gratitude and thanks to my parents, En. Ebas Bin Hj. Jusoh and Pn. Salmiah Binti Chacini for all the sacrifices and love to me.

Not least of all, I owe so much to my whole family, my husband and my best friends for their undying support and unwavering belief that I can achieve so much. Without all your prayers, love, help and benedictions, I would never have completed this thesis. Therefore, thank you all. May you all be blessed by Allah S.W.T in this world and the hereafter.

ABSTRACT

A finite switchboard automaton has an explicit mechanism which is switchboard that acts as a controller to predict the next input for the interaction within the systems. The classical version of the algebraic automata is a part of theoretical computer science which is not effectively reflecting the practical demands of the computation at the algebraic level. It unable to formalize the controller to predict the flow of the next input information into a designated output. In other words, the algebraic approach is still lacking in terms of their properties. Thus, it is necessary to understand the modeling of switching and commutative mechanisms as a controller in a machine. Fuzzy set theory can be applied to solve the control problems. This research studied on how one can incorporate the fuzzy set into finite switchboard automata and develop algebraic properties. Further, the general algebraic structure such as complete residuated lattices (CRL) has been utilized to enhance the membership grade of the fuzzy finite switchboard automata (FFSA). This research also proposed a specific algorithm for FFSA by the use of CRL. In an automata theory, some machines seldom have the possibility of overlapping transitions to the same state upon the same symbol from the different current states that are called as multi-memberships. Thus, this research considers the multi-memberships in the FFSA which lead to overcome these issues by introducing the theory of the general fuzzy switchboard automata (GFSA) and investigates the topological study of GFSA with the help of switchboard subsystems. The newly defined Kuratowski fuzzy closure operation is used to establish fuzzy topology on a GFSA. Semigroup actions are closely related to automata. By extending the algebraic properties of GFSA, the General Fuzzy Switchboard Transformation Semigroup (GFSTS) has been introduced and the concept of the covering and the products are established. The objectives of this research are achieved. The properties of the switchboard automata and subsystem need to satisfy in order to make the machine well operating.

ABSTRAK

Sebuah automata papan suis keadaan terhad mempunyai mekanisme yang jelas iaitu papan suis dimana bertindak sebagai pengawal untuk meramal input seterusnya bagi berinteraksi antara sistem. Versi klasik dalam aljabar automata adalah sebahagian daripada teori komputer sains yang ia tidak dapat mencerminkan keperluan sebenar komputer pada paras aljabar secara berkesan. Ia tidak mampu untuk meformulasikan pemprosesan komutatif dan penukaran untuk meramalkan aliran maklumat input seterusnya ke output yang telah ditetapkan. Dalam kata lainnya, pendekatan aljabar masih lagi kurang dari segi sifat mereka. Justeru itu, adalah perlu untuk memahami pemodelan menukar mekanisme sebagai peranti kawalan dalam mesin. Teori set kabur boleh digunakan untuk menyelesaikan masalah kawalan. Penyelidikan ini mengkaji bagaimana sesuatu set kabur boleh digabungkan ke dalam Mesin Papan Suis Keadaan Terhad dan menghasilkan sifat aljabar. Seterusnya, struktur aljabar umum seperti kekisi rasiduated lengkap (CRL) telah di manfaatkan untuk meningkatkan nilai keahlian dalam Automata Papan Suis Keadaan Terhad Kabur (FFSA). Penyelidikan ini mencadangkan algoritma untuk FFSA dengan menggunakan CRL. Dalam teori automata, segelintir mesin jarang mempunyai kemungkinan pertindihan peralihan kepada keadaan yang sama di atas simbol yang sama daripada keadaan semasa yang berbeza yang dikenali sebagai pelbagai keahlian. Justeru itu, penyelidikan ini mempertimbangkan pelbagai keahlian dalam FFSA ke arah untuk mengatasi isu ini dengan memperkenalkan Automata Kabur Papan Suis Umum (GFSA) dan menyiasat pandangan topologi dalam GFSA dengan bantuan suis papan subsistem. Definisi terbaru Kuratowski operasi penutupan kabur telah digunakan untuk mewujudkan topologi kabur pada GFSA. Tindakan semikumpulan adalah berkait rapat dengan automata. Dengan memperluaskan sifat aljabar pada GFSA, Peralihan Semikumpulan Papan Suis Kabur Umum telah diperkenalkan dan konsep perlindungan dan hasil darab telah ditubuhkan. Objektif-objektif dalam kajian ini telah tercapai. Sifat-sifat antara automata papan suis dan subsistem perlu saling melengkap supaya mesin boleh beroperasi dengan baik.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS AND ABBREVIATIONS

$\tilde{\delta}$	-	The augmented transition function,
δ	-	Transition function
$F_{\sim}^{*}$	-	Equivalence relation in general fuzzy automata
F	-	Final state
$\tilde{F}$	-	General fuzzy automata
$\tilde{F}^{*}$	-	General fuzzy finite switchboard automata
F_1	-	The membership assignment function,
F_2	-	Multi-membership resolution function.
$\mathcal{L}$	-	Residuated lattices
λ	-	Empty word in X^*
M_i	-	Finite switchboard state machine
$\mathcal{M}_i$	-	Fuzzy finite switchboard state machine
μ	-	Fuzzy subset
μ_A	-	Grade of membership of element of X in A
μ^c	-	Complement of μ
$\emptyset$	-	Null set
η	-	Identity map on $Q_1 \times Q_2$, surjective partial function
ω	-	The non-fuzzy output function,
ϖ	-	Restricted cascade product
Q	-	Non-empty set of states
Q_{act}	-	Active state
q_0	-	Initial state
$\tilde{R}$	-	Set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$,
ρ	-	Operation transformation
S	-	Finite semigroup
$S(\tilde{F})$	-	Finite semigroup of $\tilde{F}$

σ	-	Input symbols
$\sum$	-	Finite set of input symbols, $\sum = \{a_1, a_2, \cdots, a_m\}$,
T	-	Finite set
TS	-	Transformation semigroup
$TS(\tilde{F})$	-	Fuzzy transformation semigroup
τ	-	Fuzzy set of terminal state
ξ	-	Identity map on X
v_i	-	Produce membership value
X	-	Finite non-empty set
X^*	-	Set of all word of element of X of finite length
Z	-	Finite set of output symbols, $Z = \{b_1, b_2, \cdots, b_k\}$,
CRL	-	Complete Residuated Lattices
DFA	-	Deterministic Fuzzy Automata
FFSA	-	Fuzzy Finite Switchboard Automata
FSM	-	Finite State Machine
FSSM	-	Finite Switchboard State Machine
GFA	-	General Fuzzy Automata
GFSA	-	General Fuzzy Switchboard Automata
GFSPS	-	General Fuzzy Switchboard Poly-transformation Semigroup
GFSTS	-	General Fuzzy Switchboard Transformation Semigroup
NFA	-	Nondeterministic Fuzzy Automata

LIST OF PUBLICATIONS

1 Ebas, N. A., Amir Hamzah, N. S., Kavikumar, J., Othman, F. S., Rusiman, M. S., Ahmad, N., and Abdul-Kahar, R., Algebraic properties of finite switchboard state machine, *AIP Conference Proceedings 1974*, 2018, from doi: doi.org/10.1063/1.5041589.

2 Ebas, N. A., Amir Hamzah, N. A., Kavikumar, J., and Rusiman, M. S., Fuzzy finite switchboard automata with complete residuated lattices, *International Journal of Engineering & Technology*, 2018, 7(4): 160 – 169.

3 Kavikumar, J., Tiwari, S. P., Ebas, N. A., Amir Hamzah, N. S., General fuzzy finite switchboard automata, *New Mathematics and Natural Computation*, 2019, 15(2): 283 – 305.

CHAPTER 1

INTRODUCTION

1.1 Research background

Theoretical computer science uses models and analysis to examine computers and computation. It covers many areas of computer science to develop models and methods of analysis. Automata are one of the fundamental and significant theories in computer science. Therefore, the algebraic and topological techniques used to learn the structure of automata have been substantial. The significance of investigating finite automata or finite state machine is to reduce the gap between the precision of formal languages and the imprecision of natural languages.

In 1982, Holcombe studies the algebraic automata theory since algebra has broadly developed in many different directions (Holcombe, 1982). In the area of mathematics, abstract algebra mostly studied the algebraic structures. Most algebraic structures have more than one operation which required satisfying certain axioms. Semigroups, groups, rings and fields are examples of algebraic structures. In the concepts of science and computation, the notion of change (transition) in which a system goes from one state to another state due to internal process at various time-scales or other external manipulation. Therefore, transformations of a finite set of states fulfill this concept. Transformation semigroup defined all the different ways of set transformations that can be combined in time. Algebraic products are considered in order to manufacture an automaton out of the existing automata.

Zadeh (1965) is the first researcher who introduced the concept of a fuzzy set. In the late 1960s, the concept of fuzzy automata was introduced by (Santos, 1968) and (Wee, 1969). Fuzzy finite automata may be considered as an extended model of finite automata that includes notions like “vagueness” and “imprecision” since finite automata constitute a mathematical model of computation. After that, fuzzy automata were introduced and studied by several researchers like (Li and Pedrycz, 2005), (Malik et al., 1997), (Jin et al., 2013) and many more. Finite-state automata are the mathematical models to recognize formal languages in the theory of classical computation, and the former proposed fuzzy automata with membership values in the unit interval [0, 1] with the max-min composition (Li and Pedrycz, 2005). In general, fuzzy finite state machine (FFSM) or fuzzy finite automata (FFA) has membership grades in an interval [0,1]. However, there is a possibility to extend the membership values into more general algebraic structures. In the years 2001 and 2002, Qiu studied those so-called theories and their characterizations where he considered its membership grades under the fact of complete residuated lattices (Qiu, 2002). In 2006, Qiu extended his research into a specific type of automata that are pushdown automata, turning machine and reduction and minimization (Qiu, 2006). The study in the context of fuzzy finite automata over complete distributive lattices was done by Belohlavek (2002). Meanwhile in 2005, Li and Pedrycz studied fuzzy finite automata over lattice-ordered monoid (Li and Pedrycz, 2005).

Generally, fuzzy automata provide a systematic way of generalizing discrete applications where they can create capabilities that are hardly achievable by other tools (Pedrycz and Gacek, 2001). It provides a systematic approach to incorporating approximate reasoning into systems in the way humans do (Zadeh, 1971). Therefore, Doostfatemeh and Kremer (2005) introduced a new general definition of fuzzy automata to establish a better ground of automata and fundamental for the forthcoming applications. Furthermore, general fuzzy automata can also remove the burden of generating deterministic acceptor to calculate membership values of the strings without developing a deterministic Moore automaton and it can also be used for large fuzzy grammars and languages (Doostfetemeh and Kremer, 2006).

Sato and Kuroki (2002) introduced the notion of finite switchboard state machine that is another extension of the finite state machine/finite automata. The fundamental goal of this work was to propose an efficient algebraic technique to study finite switchboard automata. It is necessary to understand the significance of modeling of switching mechanism as a control device for any electronic system. In the same year, Inagaki (2002) used Genetic Algorithm for generating more complex deterministic finite automata (DFA) through the use of switching device to make a correct predictions on the next input symbol. It is principally motivated by the work of Madison and Malik (2002) such as decomposition, submachines, retrievability, separability, connectivity and subsystems. In the same year, Sato and Kuroki (2002) introduced the concept of fuzzy finite switchboard state machines and fuzzy switchboard transformation semigroups. The idea of switching homomorphism is introduced.

A finite switchboard state machine or finite switchboard automata is binding the concept of switching state machine and commutative state machine. However, the algebraic and topological approach of a finite switchboard automata is still lacking. Thus, this research aims to study some of its algebraic and topological properties of the above-said problem. Further, this research considers enhancing the membership value of the fuzzy switchboard state machine by the use of complete residuated lattice (CRL). It also offers the general algebraic structures associated with several important logics. According to literature, the CRL has not been applied to fuzzy finite switchboard automata (FFSA). Therefore, in this research, the theory of FFSA is extended to a more comprehensive structure by considering the membership values in a CRL. Furthermore, this research considers the multi-membership value of the General Fuzzy Switchboard Automata (GFSA). Some topological properties of GFSA are also discussed in this research.

1.2 Mathematical preliminaries and notations

This section presents some general definitions and introductions which are later used in this research.

1.2.1 Sets

A set is a collection or group of numbers or objects, considered as an object in its own right (Sharma, 2004). In a set, each object or number is called as a member or element of the set. For instance, if A is an element of B, then the notion $A \in B$ is used. The set with no elements is called as null set or empty set, $\emptyset$. The cardinality is the number of elements in a set. Let $|S|$ denotes the cardinality of S. If $|S| < \infty$, then S is called finite set otherwise it is infinite set. Set theory is fundamental to all of mathematics and it is closely connected with symbolic logic. Let A be a subset of S

denoted as $A \subseteq S$, if all the elements of the set A are also the element of set S. Meanwhile, if $A \subseteq S$ and also A is not equal to S, then A is said to be proper subset of S, the notion $A \subset S$ is used for proper subset.

The set is described in these following manner where the notation of set S given as

$$A = \{x \in S | P(x)\}$$

or

$$A = \{x | x \in S, P(x)\}$$

where x is the element of S and x satisfies the property P. Other than that, sets can also be joined or combined in several ways.

Definition 1.0: (Moderson & Malik, 2002)
Union of the sets A and B, written as $A \cup B$ is the set of all members of A or B or both. The sets defined as

$$A \cup B = \{x | x \in A \; or \; x \in B\}$$

Definition 1.1: (Moderson & Malik, 2002)
Intersection of two sets which are sets A and B, denoted $A \cap B$ is the sets of all objects that only the members of both A and B. The sets defined as

$$A \cap B = \{x | x \in A \; and \; x \in B\}$$

Definition 1.2: (Moderson & Malik, 2002)
Set difference of A and B, denoted $A \backslash B$ is the set of all the members of A that are not the members of B. When B is the subset of A, $A \backslash B$ is also known as the complement of B in A and the notion used is B^c instead of $A \backslash B$.

$$A \backslash B = \{x | x \in A \; but \; x \notin B\}$$

Definition 1.3: (Moderson & Malik, 2002)
Power set of a set A, written 2^n, which is n-tuples of elements from A where $n \in N$, is the sets of all objects that are members with all possible subsets of A.

1.2.2 Fuzzy sets

According to Zadeh (1965), the fuzzy set is a class of objects with the sets of real numbers that consist of the grade of membership that assigns from the interval [0, 1]. The notions of the fuzzy set such as inclusion, union, intersection, and complement were introduced by Zadeh (1965). Regarding the introduction of the fuzzy set, it takes this field to become wider and gave an idea as well as the knowledge to the researchers who studied in this area. This introduction is used to deal with the concept of uncertainty.

Definition 1.4: (Zadeh, 1965)
A fuzzy subset μ of X is a function of X into the closed interval [0,1].
Assume μ as a fuzzy subset of a set X where $x \in X$, $\mu(x)$ is the degree of membership also known as membership value of x in μ. Sometimes the notion μ_A for a fuzzy subset of X is used instead of μ. Let A represents as a fuzzy set and μ_A gives the grade of membership of elements of X in A.

Definition 1.5: (Moderson & Malik, 2002)

i. The support of a fuzzy subset X, μ is a crisp set defined by

$$supp(\mu) = \{x \in X | \mu(x) > 0\}$$

ii. The core of a fuzzy subset X, μ is a crisp set defined by

$$core(\mu) = \{x \in X | \mu(x) = 1\}$$

Definition 1.6: (Moderson & Malik, 2002)
Let μ and s be fuzzy subsets of a set X. For all $x \in X$ where μ^c is called the complement of μ, $\mu \cup s$ and $\mu \cap s$ is called the union and the intersection of μ and s respectively. μ^c, $\mu \cup s$ and $\mu \cap s$ is defined as follows:

$$\mu^c(x) = 1 - \mu(x)$$
$$(\mu \cup s)(\mathrm{x}) = \max\{\mu(\mathrm{x}), \mathrm{s}(\mathrm{x})\}$$
$$(\mu \cap s)(\mathrm{x}) = \min\{\mu(\mathrm{x}), \mathrm{s}(\mathrm{x})\}$$

Besides that, sometimes the symbol $\vee$ is used to denote as max and supremum meanwhile the symbol $\wedge$ is used as min and infimum. For instance, these symbols used as $(\mu \cup s\ )(x) = \mu(x)\ \vee s(x), and\ (\ \mu \cap s\ \ )(x) = \mu(x)\ \wedge s(x)\ \forall x \in X$.

1.2.3 Group theory

Group theory studies the algebraic structures known as groups. A group is a monoid with an inverse element. A monoid is a semigroup with an identity. Meanwhile, semigroup is a nonempty set with an associative binary operation. According to this general definition, semigroup is a part of group. From algebraic perspective, a semigroup action is a generalization of the notion of a group action in group theory. Semigroup action is closely related to automata where the state of the automaton and the action transformations of that state in response to inputs.

1.2.3.1 Semigroup

A semigroup generalizes the concept of a group with an associative binary operation. The function from $S \times S$ into X is called a binary operation of S where S is denoted as a nonempty set. If semigroup has an identity element, it is called as monoids. A semigroup plays an important role in fuzzy automata and fuzzy language. Mostly, the algebraic approach to automata theory relies on a semigroup. Basically, a semigroup is an algebraic structure that is closed consisting of a set together with an associative binary operation, such as additive and multiplication. It is also known as associative algebraic structure. In summary, if (S, η) where S is a non-empty set S on which a binary operation η is associative whether multiplication (*) or addictive (+), it is called a semigroup.

Associativity means that the product of elements is straightforward because it does not matter on how to evaluate the products as long as the order is maintained. One of the important associatives is for any element a of a semigroup which is n^{th} power, notation a^n is equal with the product of a with itself n times. Let $*$ be a binary associative operation and S be a nonempty set, then a pair $(S,*)$ is called a mathematical system. For instance, $(a * b) * c = a * (b * c)$ the familiar associative law of elementary algebra for $\forall\ a, b, c \in S$ then $*$ is associative and $(S,*)$ is a semigroup. If a semigroup S has the property that, $\forall a, b \in S, ab = ba$, then it is a commutative semigroup.

1.2.4 Finite State Machine

The theory of automata has been growing along together with the mathematical theory. One of them is the algebraic automata theory. The theory of machines has been applied to many fields, such as computer systems, linguistics, biochemistry and many more.

Since the number of distinguishable situations in the state machine is finite, that means the number of states is finite as well. A Finite State Machine (FSM) also known as Finite Automaton (FA), introduces the concept of a state that gives information about past history. In the state machine, all states represent all possible situations and it contains a small amount of memory. This memory can make the state machine reached the present situation, although, FSM's memory is limited by the number of states it has. Therefore, FSM is studied more in the automata theory.

The changes of the states occur from time to time and the inputs will influence the outputs of the machine. This computational device came out with the input of string while the output is one of the two values that we called Accept and Reject. The FSM can change from one state to another state. The changes are called transitions.

Definition 1.7: (Finite State Machine)

Consider a state machine as a quadruple $M = (Q, \delta, \sigma, \tau)$ where Q is a non-empty set of states, $\delta : Q \times X \rightarrow Q$ is a transition function, σ is called input symbols,$\tau: Q \times X \times Q \rightarrow L$ is a fuzzy set of terminal sets and $q_0 \in Q$ is an initial state. $M = (Q, \delta, \sigma, \tau)$ is called complete if the partial function $\sigma, \tau: Q \times \delta \rightarrow Q$ is a function.

1.2.5 Fuzzy Finite State Machine

The concept of fuzzy sets was introduced by Zadeh (1965). Meanwhile, the concept of fuzzy finite state machines was given by Malik et al., (1994).

A fuzzy finite state machine (FFSM) is a triple $M = (Q, X, \mu)$, where Q and X is a finite nonempty sets, μ is a fuzzy subset of $Q \times X \times Q \rightarrow [0,1]$. Let X^* denote the set of all word of elements of X of finite length. Let λ be the empty word in X^* and let $|x|$ be the length of $x \in X^*$. $\mu^*: Q \times X^* \times Q \rightarrow [0,1]$ is defined by

$$\mu^*(q, \lambda, p) = \begin{cases} 1 \; if \; q = p \\ 0 \; if \; q \neq p \end{cases}$$

where $\lambda = xa$

$$\mu^*(q, xa, p) = \vee \{\mu^*(q, x, r) \wedge \mu(r, a, p) \,|\, r \in Q\}$$

1.2.6 Finite Switchboard State Machine

Finite Switchboard State Machine (FSSM) is extended from finite state machine. M is called a switchboard state machine if M is commutative and switching. Let $=(Q, \delta, \sigma, \tau)$, $Q = \{q, p\}, X = \{x, y\}$. If $\delta_{xy}(q, p)$ $=\delta_{yx}(q, p)$ for each $q, p \in Q$, for each $x, y \in X$ then M is commutative. $\delta_x(q, p) = \delta_x(p, q)$ for each $q, p \in Q, x \in X$ then M is called switching. Switchboard is a mechanism that is able to control the direct flow of information from one state to another state. Besides, it is also used for communication between one subsystem to another subsystem. In order to make the systems connected, the properties of that systems need to be satisfied.

1.2.7 Homomorphism

In algebra, homomorphism is a structure preserving map between two algebraic structures of the same type (Moderson & Malik, 2002). Meanwhile, in terms of topology, homomorphism can permit more kinds of transformation. Transformation semigroups are closely related to automata in finite state machine (Sato,2003). However, some properties need to be studied through the conception of homomorphism.

Definition 1.8: (Moderson & Malik, 2002)
Let $M_1 = (Q_1, \delta_1, \sigma_1, \tau_1)$ and $M_2 = (Q_2, \delta_2, \sigma_2, \tau_2)$ be a FSSM. M_1 and M_2 is a pair (α, β) of mappings $\alpha: Q_1 \rightarrow Q_2$ and $\beta: X_1 \rightarrow X_2$ is called homomorphism, written as

$(\alpha,\beta) : M_1 \to M_2$ such that $\delta_1(q,x,p) \leq \delta_2(\alpha(q),\beta(x),\alpha(p))$ for any $q,p \in Q_1$ and $x \in X_1$.

The pair (α,β) is called a strong homomorphism if

$$\delta_2\big(\alpha(q),\beta(x),\alpha(p)\big) = \vee \{\delta_1(q,x,t) | t \in Q_1, \alpha(t) = \alpha(p)\}$$

Where $\forall q,p \in Q_1$ and $\forall x \in X_1$.

To summarize, if $X_1 = X_2$ and β is the identity map, then $\alpha: M_1 \to M_2$ is a homomorphism or strong homomorphism accordingly. Next, if (α,β) is a strong homomorphism with α is one-one, then $\delta_2\big(\alpha(q),\beta(x),\alpha(p)\big) = \delta_1(q,x,p)$.

1.3 Problem statement

In the study of the algebraic properties of automata theory, the classical versions are often misunderstood to reflect the real needs of the current computer science. It faces some problems to navigate or to predict the flow of the next input information into a designated output when it receives given input information from a sequence of integers. It is unable to formalize the switching and commutative processing, which is nowadays central to the computation, whenever we need to reevaluate the global transition in the finite state machine while it was virtually unneeded in the past. In other words, the algebraic approach is still lacking in terms of their properties which is the properties regarding the switchboard state machine. Thus, it is necessary to understand the modeling of switching mechanisms as a control device.

A switchboard state machine is a special kind of finite automata. It has an explicit mechanism in choosing another state that acts as a controller which is called the switchboard. However, the switchboard does not attempt to predict the next state input. Instead, it performs a correct prediction of the next input for the interaction between the subsystems. Therefore, the switchboard state machine is used for communicating between one subsystem to another subsystem of the whole system and it also maintains the mapping between objects within the subsystems.

Fuzzy Finite Switchboard State Machine (FFSSM) was introduced and studied by (Sato and Kuroki, 2002). They were focusing on the properties of homomorphism, semigroup, transformation semigroup and algebraic product in

FFSSM. However, the properties regarding the switchboard and subsystems are still lacking. Thus, one of the purpose of this study is to enhance the algebraic properties in Fuzzy Finite Switchboard Automata. Then, this research introduced the fuzzy finite switchboard subsystems which form as L-sublattices

In addition, the fuzzy set theory is widely used in solving control problems and it can enhance the algebraic properties by cooperating finite switchboard automata with fuzzy. Complete Residuated Lattices (CRL) is used as the structure of membership values. Hence, it is necessary to study the algebraic properties of the finite switchboard automata by applying CRL. Moreover, on that point, some possibilities of topological concepts are available for finite switchboard automata. In order to construct the machines to operate functionally, the concept of topology for the machine needs to be studied.

In DFA, a string is either accepted or rejected. In fuzzy automata, acceptance or rejection of the string is not the only issue. Computing the membership value of the acceptable string that describes the fuzziness is also an issue. Besides, there is some possibility of multi-membership at the same time in an active state. According to Doorstfatemeh and Kremer (2006), some researchers tried to determine the membership of the string by developing a deterministic automaton. However, it becomes impractical for large fuzzy grammar and languages. Thus, GFA is used to analyze the membership value of the acceptance strings by using fuzzy set operations and removes the burden of generating deterministic acceptors and resolves multi-membership. Therefore, inspired by Doorstfatemeh and Kremer's (2006) idea, this research introduced the concept of General Fuzzy Switchboard Automata (GFSA) and also proposed an efficient algebraic technique to study the GFSA. Since semigroup is an important algebraic approach in the automata theory, thus it is necessary to study semigroup in GFSA. By extending the algebraic properties of GFSA, the General Fuzzy Switchboard Transformation Semigroup (GFSTS) is introduced.

1.4 Research objectives

The objectives of the research are as follows.

1) To derive efficient algebraic properties of the fuzzy finite switchboard automata

2) To develop algebraic and topological properties of new fuzzy automata namely general fuzzy switchboard automata (GFSA).
3) To extend the algebraic properties on GFSA to the General Fuzzy Switchboard Transformation Semigroup (GFSTS).

1.5 Scope of study

This research is limited to a finite state machine which means finite number of possible states. There are two types of finite state machines that are deterministic finite automata (DFA) and non-deterministic finite automata (NFA). For DFA it can only be one state transition at a time. Meanwhile, NFA can have many possible states transitions at once for every input symbol-state pair.

1.6 Significance of study

This research offers the foundations of the applications of fuzzy finite automata theory in the algebraic and topological aspects in more detail. The application of this theoretical account is its potential to solve problems in learning systems, pattern recognition and database theory. Moreover, the automata can recognize more extensive classes of formal languages. It can also recognize fuzzy languages from the viewpoint of level structures.

The present research is more significant due to the vast applications in real-world problems involving uncertainties. For example, the fuzzified algebraic automata structures have become the best tool in the fields of control engineering, computer science and automata theory. It is hoped that the present work has great potential in a broad range of social and economic problems, such as cytological image analysis, energy consumption and cryptography.

1.7 Outline of thesis

This thesis intends to study the algebraic and topological properties of finite switchboard state machine by using complete residuated lattices and multi-membership resolution algorithm. In order to help organize any thoughts or arguments made along this research and for better understanding, the overview of the main content in this research is provided. By including the introductory chapter, there are seven chapters in this thesis. Each of the main chapter consists of several sub-chapters to make it more systematic and understandable.

Chapter 1 is an introduction for the automata theory, finite state machine and some basic definitions on the properties used. The problem statement discussed on the assumption and the idea to further studies about the problem based on the existing research. Other than that, the objectives of the research, the scope of the study, the significance of the study, the outline of the thesis and the overview of the methodology for the whole research are also discussed.

Chapter 2 discussed the existing researches that are related to this thesis. Many researchers studied the automata theory by using different approaches. Despite the long history of fuzzy automata and automata theory, there are still some issues that need some kind of improvement. By referring to previous researches, the enhancements are sought.

Chapter 3 discussed the methods, the properties and the algorithms in detail by referring to the objectives of the research.

Chapter 4 provides the efficient algebraic properties in the finite switchboard state machine with some real-life applications. In order to obtain the optimal membership value, CRL is applied and the calculations are provided. Thus, the first objective of this research is achieved.

Chapter 5 offers the solution of the second objective that is to develop algebraic and topological properties of new fuzzy automata namely general fuzzy switchboard automata (GFSA). Some properties and the application of GFSA are discussed and defined. The examples and the calculations of GFSA are shown.

The third objective is discussed in Chapter 6. Semigroup actions are closely related to automata. It is easier and simple to study the properties of the system in a small part. Further from GFSA, the General Fuzzy Switchboard Transformation Semigroup (GFSTS) is introduced. The structures of GFSTS are examined through products and covers. Some related properties are shown in this chapter.

Last but not least, Chapter 7 is the conclusion of the whole research. The results and recommendations are also discussed.

CHAPTER 2

LITERATURE REVIEW

2.1. Introduction

This chapter reviews existing researches that are related to finite switchboard state machine, residuated lattices and general fuzzy automata. The knowledge shared some ideas to understand the research problems. The concept of a fuzzy set was introduced by Zadeh in 1965. In the following years, many researchers extended his studies by referring to the concept of a fuzzy set applying to different contexts.

Automata are self-acting machines that able to understand and perform computations through a series of states of action and events. At each state of the computation, a transition function plays an important role to determine the next event based on a finite portion of the current state. As a result, it accepts the input once the computation reaches an accepting state. A finite automaton establishes a useful mathematical model in theoretical computer science. Finite state machines (FSM) are ideal computational models for a small amount of memory. However, the memory cannot be maintained in FSM. This mathematical model of a machine can only reach a finite number of states and transitions between these states.

The main application of FSM is in mathematical problem analysis. Based on the idea from Zadeh, Wee (1967) introduced the mathematical formulation of fuzzy automata. Sato and Kuroki (2002) extended the study from Wee (1967) and introduced the notion of the finite switchboard state machine. However, they were only focusing on homomorphism, semigroup, transformation semigroup and algebraic product in Fuzzy Finite Switchboard State Machine (FFSSM). The properties are still lacking in the subsystem and the switchboard state machine or known as the switchboard automata. Based on the idea from Sato and Kuroki (2002), it motivates further investigation on the algebraic and topological studies of finite switchboard state machines since the algebraic techniques are still lacking (Sato & Kuroki, 2002).

2.2. Existing researches

The finite state machine is also known as finite automata. There are two types of finite state machines that are deterministic finite state machines, or deterministic finite automata and nondeterministic finite state machines or non-deterministic finite automata. Both are slightly different in ways the state machine represented visually. However, the ideas are still based on computational ideas. For deterministic finite automata, it can only be one state transition at a time. Meanwhile, nondeterministic finite automata can have many possible states transitions at once for every input symbol-state pair. Finite state machines involved a mathematical model of computation that can be used to simulate sequential logic and some computer programs. The table below are the existing researches related to finite state machines, fuzzy finite state machines, finite switchboard state machines, complete residuated lattices and general fuzzy automata from year 1959 until year 2019.

Table 2.1: Finite state machines

Author;s name	year	title	Description
Rabin and Scott	1959	Finite automata and their decision problems	Introduced nondeterministic finite automata, and provided another canonization method to finite state machines
Alur and Dill	1994	A theory of timed automata	Introduced and studied a new case of automata called timed (finite) automata that model the behavior of real-systems over time. The finite automaton accepts timed words while the infinite sequences are associated with each symbol in which a real-valued time of occurrence
Broy and Wirsing	2000	Algebraic state machines	Introduced the concept of the algebraic state machine as a state transition system. A notion of object-oriented component is introduced and the methodologies of the algebraic state machines which can formalize such components are shown.
Glabbeek and Ploeger	2008	Five determinization algorithm	Presented several determinization algorithms, all variants of the well-known subset construction to reduce memory usage and produced smaller output automata.

Table 2.1 (continued)

Author;s name	Year	Title	Description
Meduna and Zemek	2012	Jumping finite automata	Investigated a new type of automata found in jumping finite automata which work like classical finite automata except that they read an input word discontinuously which means that, they can jump over some symbols within the words and continue their computation after reading a symbol.
Ather et al.,	2013	An algorithm to design finite automata that accept strings over input symbol a and b having exactly x number of a & y number of b	Come up with the idea to develop an algorithm to design finite automata that accept strings over input symbol a, b having exactly x number of a and y number of b.
Kavikumar et al.,	2013	*N*-structures applied to finite state machines	Introduced the notion of an *N*-finite state machine where the *N*-structure (negative-valued function) applied to finite state machines. A condition for an *N*-finite state machine to satisfy the *N*-exchange property is established.

A fuzzy finite state machine is specialized from a finite state machine. The main difference, between the finite state machine and the fuzzy finite state machine, is that with input the former can move from one state to another state (deterministic or non-deterministic) whereas the latter may move to many states. According to a certain possibility degree that is membership value, the fuzzy automaton can be changed from one state to another state.

Automata is the best tool for handling general computational systems over discrete spaces and combining the structure of fuzzy logic and automata theory. The outcome is the fuzzy automata, which can handle continuous spaces. The study of fuzzy automata was initiated by Santos (1968) and Wee (1967) after Zadeh (1965) introduced the fuzzy set theory. A formulation of fuzzy automata and its application as a model of learning systems had been discussed by Wee and Fu (1969). This article was based on the concept of fuzzy sets introduced by Zadeh, meanwhile the Mealy's formulation of finite automata was formulated in a class of fuzzy automata.

Malik et al., (1997) introduced the products of a fuzzy finite state machine which are coverings, cascade, wreath product and fuzzy transformation semigroups. They also introduced the notion of polysemigroups and weak coverings to overcome the difficulties that arise from fuzzification. Jin et al., (2013) studied the algebraic properties of fuzzy finite automata after Li and Pedrycz (2005) introduced the fuzzy automata theory on lattice-ordered monoids. In 2016, Sharma et al., studied algebraic of fuzzy multiset finite automata and introduced the concepts of homomorphism, coverings of fuzzy multiset finite automata and fuzzy multiset transformation semigroup. Table 2.2 represents the existing researches related to fuzzy finite state machines from the year 2010 until the year 2017.

Table 2.2: Fuzzy finite state machines

Author's name	Year	Title	Description
Ignjatovic et al.,	2010	Myhill–Nerode type theory for fuzzy languages and automata	Developed a general Myhill–Nerode type theory for fuzzy languages that consists of the membership values in an arbitrary set with two elements that are 0 and 1. Then, they introduced and studied Nerode's and Myhill's automata to a fuzzy automaton with truth values in a complete residuated lattice.
Jun and Kavikumar	2011	The bipolar fuzzy finite state machine	The notion of a bipolar fuzzy finite state machine is introduced with the basis of a fuzzy finite state machine. The related properties and the condition for a bipolar fuzzy finite state machine to satisfy the bipolar exchange property is established and discussed
Ciric et al.,	2012	Bisimulations for fuzzy automata	Investigated the concept of bisimulation for fuzzy automata since bisimulation can reduce the number of states. They studied the equivalence between fuzzy automata and gave a comprehensive overview of various concepts related to the bisimulations.
Subramaniyan and Rajasekar	2012	Homomorphism in bipolar fuzzy finite state machines	They introduced homomorphism, strong homomorphism in bipolar fuzzy finite state machines and discussed their properties using the bipolar-valued fuzzy set.
Jancic and Ciric	2014	Brzozowski type determinization of fuzzy automata	This Brzozowski determinization method gives the best results for the determinization of fuzzy automata for finite cases. For nondeterministic automata, Brzozowski type determinization of fuzzy automaton results in minimal crisp-deterministic fuzzy automaton that is equivalent to starting fuzzy automaton.
Micic et al.,	2015	Determinization of fuzzy automata by means of the degree of language inclusion	It mentioned the canonization method is the significant determinization method that provided minimal crisp deterministic fuzzy automata which is equivalent to the original fuzzy finite automaton. Both bisimulation and Brozozowski determinization are methods that can reduce the number of states.
Jancic et al.,	2016	Further improvements of determinization methods for fuzzy finite automata	They produced smaller automata but required the same computation time. The algorithms that they create, can be used in the Brzozowski type algorithm and also improve its performance.

Table 2.2 (continued)

Author's name	Year	Title	Description
Abdullah et al.,	2017	Cubic finite state machine and cubic transformation semigroups	Provided a new generalization of fuzzy finite state machines, fuzzy transformation semigroup and its relationship with considering the cubic structure and its properties are defined.
Khamirrudin et al.,	2017	decomposition of bipolar fuzzy finite state machines and transformation semigroup	The concepts of decomposition of fuzzy finite state machines and fuzzy transformation semigroups have been generalized. They were substituting the interval as the truth structure of the transition function in bipolar setting for the study of algebraic automata.
Sing et al.,	2017	On algebraic study of type -2 fuzzy finite state automata	Introduced the concept of automata theory in a type-2 fuzzy set and discussed the concept of homomorphism, transformation semigroup and direct product for type-2 fuzzy finite state automata.

Even though the researchers came up with different approaches in a fuzzy state machine, most of them studied the properties of transformation semigroup. Hence, it is clear that the algebraic automata theory is an appropriate approach to study the concept of a finite semigroup that is important in automata theory.

In 2002, Sato and Kuroki introduced the notion of a finite switchboard state machine that is another extension of the fuzzy state machine. According to Sato (2002), the switchboard state machine can be defined by binding the concept of switching state machine and commutative state machine together. It is a special kind of finite automata and has an explicit mechanism in choosing another state that acts as a controller. However, the properties of FSSM are still lacking in terms of the connection between the switchboard and the subsystem.

The introduction of topology in automata theory is not only a powerful tool for studying varieties. It also led to unexpected developments, which are very useful for the current research in this area. In the year 2005, Jun introduced the concept of intuitionistic fuzzy finite state machines by using the notion of intuitionistic fuzzy sets. In 2006, he furthered his research in the intuitionistic fuzzy finite switchboard state machines. He introduced the notion of intuitionistic fuzzy finite switchboard state machines and strong homomorphisms of the intuitionistic fuzzy finite state machine. The related properties are identified and shown that the family of equivalence classes is a finite semigroup with identity.

The bipolar fuzzy finite state machine was discussed by Jun and Kavikumar (2011). In this research, the notion of a bipolar fuzzy finite state machine is introduced with the basis of a fuzzy finite state machine. There are some related properties and the condition for a bipolar fuzzy finite state machine to satisfy the bipolar exchange property is established and discussed. Kavikumar et al., (2012) extended their research from bipolar fuzzy finite state machine to bipolar fuzzy finite switchboard state machine. They introduced and investigated related properties of bipolar fuzzy finite switchboard state machines. The concepts of bipolar submachine, the notion of the bipolar-valued fuzzy finite state machine, bipolar connected and bipolar retrievable have been established. Table 2.3 represents the current existing researches related to FSSM in the year 2016 until the year 2019.

Table 2.3: Finite switchboard state machines

Author’s name	Year	Title	Description
Mahmood and Khan	2016	Interval neutroshopic finite switchboard state machines	Introduced the concept of interval neutrosophic finite state machine, interval neutrosophic finite switchboard state machine using the notion of interval neutrosophic set. The concepts of homomorphism and strong homomorphism of interval neutrosophic finite state machines are established
Khan et al.,	2018	Single valued neutrosophic finite state machine and switchboard state machine	The notion of single valued neutrosophic finite state machine, single valued neutrosophic successor, single valued neutrosophic subsystem, single valued submachine, single valued neutrosophic switchboard state machine, homomorphism and strong homomorphism between single valued neutrosophic switchboard state machine are introduced.
Kavikumar et al.,	2019	Restricted cascade and wreath products of fuzzy finite switchboard state machines	Continued the study of the fuzzy finite switchboard state machine by examining the direct product, Cartesian product, the covering and other products to propose an efficient algebraic. In addition, the perfect switchboard machine is introduced and the relations among the products also are examined
Reena	2019	Bipolar vague finite switchboard state machine	Bipolar vague FSSM is introduced and the related properties such as bipolar switching, bipolar homomorphism, bipolar strong homomorphism and bipolar retrievable are investigated.
Yaqoob and Abughazalah	2019	Finite switchboard state machine based on cubic sets	The notion(strong and good) of subsystem of cubic FSSM are introduced and the related properties are investigated.

For a given automaton with an input alphabet, each finite word will induce an operator, which is from a finite monoid under composition on the set of states of automaton. The structure of subclasses of the regular languages can be obtained by using this algebraic perspective. This approach is known as the Algebraic Automata Theory. Regarding the existing research, it can be seen that many researchers using complete residuated lattice in fuzzy automata since this method consists of five algebras that are BL-algebra, Heyting algebra, Gödel algebra, Product algebra and MV-algebras. Complete residuated lattice can reduce the growth of the number of states during the determinization.

Some properties of residuated lattices were written by Belohlavek (2003) to investigate some properties of residuated lattices algebras which are universal to determine the structure of the membership value of various systems of fuzzy logic. Stanovsky (2007) studied the commutative idempotent residuated lattices. He investigated the variety of residuated lattices with a commutative and idempotent monoid reduct. Other than that, the structure of residuated lattices was discussed by Blount and Tsinakis (2003). This research established general structural theory for the class residuated lattices as a whole. It also developed the notion of a normal subalgebra and showed residuated lattices is an ideal variety in the sense of an equation class in which congruence corresponds to normal subalgebra in the same way that ring congruence corresponds to ring ideals. Jasem and Bratislava (2007) studied on ideals of lattice ordered monoids. They introduced the notion of an ideal of a lattice ordered monoid A, meanwhile, the relations between ideals of A and congruence relations on A are investigated.

Automata theory over complete residuated lattice-valued logic is known as L-valued automata. It has been proposed by Qiu (2001). Based on the idea from Qiu, Xing and Qiu (2009) studied the categorical issues of L-Valued Automata (L-VAs), also known as automata theory according to the complete residuated lattice-value logic. They investigated the relationship between the categories, the existence of isomorphisms between the categories and specific relationships between the output L-valued subsets of generalized L-VAs and the output L-valued subsets of NDAs. In the same year, Xing et al., (2009) discussed some properties of L-valued context-free grammars, languages, and pushdown automata then proved the equivalence between L-valued context-free grammars and L-valued pushdown automata. Guo (2012) studied the pushdown automata in L-Vas and realized that the definition of the L-

valued Chomsky Normal Form in Xing et al., (2009) was slightly different from the definition in Xing and Qiu (2009). Therefore, he introduced a more general *L*-valued Chomsky Normal Form to unify the two definitions. Besides, he also showed that for an *L*-valued context-free grammar and an *L*-valued Greibach Normal Form can be equivalently constructed.

Wu and Qiu (2010) defined a kind of Mealy type of *L*-VFAs (MLFAs), a generalization of *L*-VFAs, *L*-valued languages (*L*-VLs) and *L*-valued regular languages (*L*-VRLs) recognized by *L*-VFAs, and provide some properties of *L*-VRLs. They introduced two kinds of state-wise equivalence relations and a minimization algorithm of the MLFAs and *L*-VFAs as well. Wu et al., (2012), the theory of turning machines was established based on complete residuated lattice-valued logic where it is a continuation of *L*-VAs.

Table 2.4: Complete residuated lattices

Author’s name	Year	Title	Description
Ignjatovic et al.,	2008	Determinization of fuzzy automata with membership values in complete residuated lattices	Introduced a new method for determinization of fuzzy finite automata with membership values in complete residuated lattices and compared with the previous method. The method introduced by ignjatovic always gives smaller automaton.
Li	2011	Finite automata theory with membership values in lattices	Lattice-valued finite automata is a common generalization of fuzzy automata and weighted automata. He also introduced the technique of extended subset construction and gave a minimal algorithm of lattice-valued deterministic finite automata.
Ghorani and Zahedi	2012	Characterizations of complete residuated lattice-valued finite tree automata	Studied the characterization of complete residuated lattice-valued finite tree automata. The *l*-valued regular tree language is defined and the minimization of the algorithm of the *l*-fta is presented
Busneag and Piciu	2015	A new approach for classification of filters in residuated lattices	Proposed a new approach for the study of the classification of filters in residuated lattices. This idea comes up since there are multiplicities of the name of the filters that make it difficult to study and do connections between them.
Pan et al	2017	Nondeterministic fuzzy automata with membership values in complete residuated lattices	They introduced two language-equivalent relations and their fuzzy versions. In addition, a new kind of nondeterministic fuzzy automata is presented.

Table 2.4 (continued)

Author's name	Year	Title	Description
Konecny and Krupka	2017	Complete relations on fuzzy complete lattices	Generalized the notion of a complete binary relation on the complete lattice to residuated lattice valued ordered sets and the properties are shown. By focusing on complete fuzzy tolerances on fuzzy complete lattices, one-to-one correspondence with extensive isotone galois connections and any fuzzy complete lattice factorized by a complete fuzzy tolerance is again a fuzzy complete lattice is proven.
Gautam et al.,	2018	Categories of automata and languages based on a complete residuated lattice	A new category of fuzzy automata based on CRL is introduced. The categorical concepts such as product, equalizer and their duals in this category are studied.

In the year 2005, Doostfatemeh and Kremer introduced the new direction in fuzzy automata namely as General Fuzzy Automata (GFA). GFA can model applications that have all parts of characteristics and one of the characteristics is each state can be reached which gets activated from at least an initial state, according to the sequences of input symbols (Doorstfatemeh and Kremer, 2005). Compared to the conventional Fuzzy Finite Automata, GFA is a more application-friendly tool. In the same year, Doostfatemeh and Kremer (2005) introduced a new general formulation of fuzzy automata with outputs. They showed that by using Moore and Mealy models in fuzzy automata, the output mapping can be handled. The algorithm is developed to convert different models of fuzzy automata to each other. Regarding the Deterministic Fuzzy Automata (DFA), some of the researchers tried to calculate the membership value of the strings by developing the deterministic Moore automaton. Despite that, Doostfatemeh and Kremer (2006) showed the newly developed paradigm of GFA to solve the problem by removing the burden of generating deterministic acceptors to calculate the membership value of the acceptable string by using fuzzy set operations without developing a deterministic Moore automaton. They solved the problems related to the larger fuzzy grammar and languages by using GFA. Based on the work from Doostfatemeh and Kremer, many researches extended their studies by applying GFA to different contexts.

Table 2.5: General fuzzy automata

Author's name	Year	Title	Description
Horry and Zahedi	2009	Uniform and semi-uniform topology on general fuzzy automata	Defined the concept of uniform and semi-uniform topology on GFA and also discussed the properties. Some researchers studied about basic logic (BL) into GFA namely as BL-GFA.
Abolpour and Zahedi	2012	Isomorphism between two BL- general fuzzy automata	Focused on the behavior of BL-GFA and proved that the minimal reduction of the reachable part of a BL-GFA is the minimal realization of the behavior.
Shamsizadeh et al.,	2016	Bisimulation for BL-general fuzzy automata	They defined bisimulation for BL-GFA and developed an algorithm for two given BL-GFA in order to determine bisimulation between them.
Honry and Zahedi	2013	Some (fuzzy) topologies on general fuzzy automata	Studied some (fuzzy) topologies on GFA where some Lowen-type and Chang type fuzzy topologies structure on GFA is introduced and different types of fuzzy topologies are obtained by presenting some notions and theorems.
Shamsizadeh and Zahedi	2016	Intuitionistic general fuzzy automata	Defined the concept of intuitionistic general fuzzy automaton (IGFA), max–min IGFA, admissible relation for the IGFA, admissible partition for the IGFA, quotient IGFA and language for the IGFA by considering the notions of GFA.
Shamsizadeh and Zahedi	2016	Minimal and statewise minimal intuitionistic general L-fuzzy automata	They show that for any (α, β)-language L, there exist a minimal intuitionistic general L-fuzzy automaton recognizing L by considering the notions of the intuitionistic general L-fuzzy automaton and (α, β) -language.
Horry	2016	Irreducibility on general fuzzy automata	Studied the covering of a max-min general fuzzy automaton, admissible partitions of a max-min general fuzzy automaton, δ-orthogonality of admissible partitions, and irreducible max-min general fuzzy automata and the relations between them.
Horry	2017	Application of a group in general fuzzy automata	He defined the concepts of fuzzy normal kernel of a general fuzzy automaton, fuzzy kernel of a GFA, adjustable, multiplicative and relationship between them.
Abolpour and Zahedi	2017	General fuzzy automata based on complete residuated lattice-valued	They discussed the relationship between the category of General Fuzzy Automata on the basis of Complete Residuated Lattice-valued (L-gfas) and the category of non-deterministic automata (ndas) and also the relationship between the output L-valued subsets of generalized L-gfas and the output L-valued subsets of ndas.

Table 2.5 (continued)

Author’s name	Year	Title	Description
Saeidi and Shamsizadeh	2019	Transformation of BL-general fuzzy automata	Prove that any BL-general fuzzy automaton (BL-GFA) and its quotient have the same behavior and the minimal quotient BL-GFA and minimal quotient transformation of the BL-GFA is obtained by considering the notion of maximal admissible partition.
Shamsizadeh and Zahedi	2019	Bisimulation of type 2 for BL-general fuzzy automata	They defined bisimulation of type 2 for a basic logic GFA and show that if there exists a bisimulation of type 2 between two basic logic GFA, then they have same behavior.

The GFA is getting more useful and popular among the researchers. Its contribution to neural networks has been considerable. GFA enhances the ground of fuzzy automata and make this appealing tool more application-friendly tool and useful. Highly motivated by the work of Sato and Kuroki (2002), this research studied and introduced the concept of general fuzzy switchboard automaton (GFSA) and proposed an efficient algebraic technique to study GFSA.

2.3. Summary

Basically, the algebraic and topological techniques are used to learn the structure of automata. Inspired by the work of Sato and Kuroki (2002), the fundamental goal of this research is to propose an efficient algebraic technique to study finite switchboard state machines since the existing research still lacking in algebraic approach and topological approach. Sato and Kuroki were only focusing on homomorphism, semigroup, transformation semigroup and algebraic product of FSSM. Since switchboard state machine is also used for communication between the subsystems, it is necessary to study the properties of connection between switchboard and subsystem. Thus, by understanding the modeling of switching mechanisms as a control device, it can enhance the algebraic and topological properties. Fuzzy Finite State Machine is able to change from one state to another state according to the membership value. CRL is used as the structure of truth-value that able to enhance the membership value from [0,1] to a more general algebraic structure. By combining these ideas, this research studies on algebraic and topological study of the finite switchboard state machine by using CRL. In addition, fuzzy finite switchboard subsystem is introduced and forms a complete *L*-sublattice. Chiefly motivated by Doorstfatemeh and Kremer (2005), the concepts of switchboard properties are studied in General Fuzzy Automata, namely, the General Fuzzy Switchboard Automata (GFSA). A semigroup is an algebraic structure that shows a very close connection between self-adjoint operators. It is partly important because they do arise in so many places. Since the semigroup is an important algebraic structure in automata theory, it is necessary to study its properties regarding GFSA by extending the algebraic properties of GFSA to General Fuzzy Switchboard Transformation Semigroup (GFSTS).

CHAPTER 3

METHODOLOGY

3.1 Introduction

This chapter describes the methodology to achieve the objectives for this study. After considering the objectives of the study, the algebraic properties are necessary to study in finite switchboard automata. The algorithm to check the validity of switchboard automata is presented. After that, the new algorithm for the Fuzzy Finite Switchboard Automata (FFSA) by using CRL has been presented in order to enhance the algebraic properties. General Fuzzy Automata (GFA) is used to resolve multi-membership since there are some problems to define membership value for the active state in the machine, if an active state has multi-membership value. Doostfatemeh and Kremer provided an algorithm for multi-membership resolution. By extending their algorithm, the new algorithm of General Fuzzy Switchboard Automata (GFSA) is presented.

3.2 Algebraic properties of Finite Switchboard Automata

The pair of algebra is to study on how the input information is transformed into the state information and the input state information into the output information (William, 1973). Algebra is well-known as a generalization of arithmetic in which letters representing numbers are combined according to the rules of arithmetic (Tabak, 2011). There is a certain theory that has to be satisfied for the algebraic properties in finite switchboard automata. A semi group is one of the properties that satisfied certain condition. In theoretical computer science, properties of languages depend on the algebraic properties of numerous transformations of semi groups related to them. The properties that fulfilled the condition are associative property and closure property. Closure property can be satisfied as a set that is closed under an operation or collection of operations.

$v + u = u + v$ (commutative)
$(v + u) + w = v + (u + w)$ (associative)
$a \cdot (v + u) = (a \cdot v) + (a \cdot u)$ (distributive)

In all the properties, $v.u.w$ are n-dimensional vectors, and a is a constant. The linear algebra is basically about linear transformations where the two operations of vector addition and scalar multiplication are linear transformation. Another important basic algebraic properties is the dot product.

$v \cdot u = u \cdot v$ (commutative)
$(u + v) \cdot w = (u \cdot w) + (v \cdot w)$ (distributive)
$(a \cdot u) \cdot v = a \cdot (u \cdot v) = u \cdot (a \cdot v)$ (associative)

3.3 Complete Residuated Lattices

An algebraic structure with strong connections to mathematical logic is known as a residuated lattice (Ignjatovic et al., 2013).

Definition 3.0: (Ignjatovic et al., 2013)
The algebra $\mathcal{L} = (L,\wedge,\vee,\otimes,\rightarrow,0,1)$should satisfy three conditions:

a) $(L,\wedge,\vee,0,1)$ is a lattice with the least element is 0 and the greatest element is 1
b) $(L,\otimes,\vee)$ is a commutative monoid with the unit 1,
c) $\otimes$ and $\rightarrow$ form an adjoint pair. For example, they satisfy the adjunction property: $\forall x, y, z \in L, x \otimes y \Leftrightarrow x \leq y \rightarrow z$.

$\mathcal{L}$ is called complete residuated lattice if $(L,\wedge,\vee,0,1)$ is a complete lattice. $\otimes$ is called a multiplication, $\rightarrow$ represents as residuum, $\wedge$ and $\vee$ is supremum and infimum respectively. Multiplication, $\otimes$ and residuum, $\rightarrow$ are planned for modeling the conjunction and implication of the corresponding logical calculus. In addition, supremum $\vee$ and infimum $\wedge$ are intended to model the general and existential quantifier.

The notion $x \Leftrightarrow y$ also known as biimplication can be written as $(x \to y) \wedge (y \to x), x \to y = \min(1 - x + y, 1)$ is a complete residuated lattice which is general algebraic structure. Meanwhile $x \otimes y = \max(x + y - 1, 0)$ is Standard Lukasiewcz algebra. Heyting algebra is a residuated lattice that satisfies $x \otimes y = x \wedge y$ and the notion of Standard Godel algebra is $x \otimes y = \min(x, y)$ and $x \to y = 1$ if $x \leq y$ and y otherwise is a Heyting algebra (Pan et al., 2008). There are some properties of complete residuated lattice in the following lemma:

Lemma 3.0: (Wu and Qiu, 2010)
Let $\mathcal{L}$ be a complete residuated lattice. Then $x, y, z \in L$ and $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \subseteq L$ the following properties hold:

1) $x \leq y$ if and only if $x \to y = 1$
2) $y \leq z \Rightarrow x \otimes y \leq x \otimes z$
3) $(x \leftrightarrow y) \otimes (y \leftrightarrow z) \leq x \leftrightarrow z$
4) $x \otimes \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \otimes y_i)$
5) $x \otimes \bigwedge_{i \in I} y_i = \bigwedge_{i \in I}(x \otimes y_i)$
6) $\bigvee_{i \in I} x_i \to y = \bigwedge_{i \in I}(x_i \to y)$
7) $x \to \bigwedge_{i \in I} y_i = \bigwedge_{i \in I}(x \to y_i)$
8) $\bigvee_{i \in I} (x \to y_i) \leq x \to \bigvee_{i \in I} y_i$
9) $\bigvee_{i \in I} (x_i \to y) \leq \bigwedge_{i \in I} x_i \to y$
10) $\bigwedge_{i \in I} (x_i \leftrightarrow y_i) \leq \bigwedge_{i \in I} x_i \leftrightarrow \bigwedge_{i \in I} y_i$
11) $\bigvee_{i \in I} (x_i \leftrightarrow y_i) \leq \bigvee_{i \in I} x_i \leftrightarrow \bigvee_{i \in I} y_i$
12) $x \leftrightarrow y \leq (x \otimes z) \leftrightarrow (y \otimes z)$.

Let $M = (Q, X, \mu)$ be a Fuzzy Finite Automata (FFA) (Sipser, 2005) where Q and X are finite non-empty sets and μ is a fuzzy subset of $\mu: Q \times X \times Q \to [0,1]$. Q is represented as the set of states, X is the set of inputs and μ is the transition function. Let X^* be the set if all word elements of X of finite length. Let β be the empty words in X^* and $|x|$ be the length of finite length. Define $\mu^*: Q \times X^* \times Q \to [0,1]$ by

$$\mu^*(q, \beta, p) = \begin{cases} 1, & q = p \\ 0, & q \neq p \end{cases}$$

and for all $b \in X^*, x \in X$,

$$\mu^*(q, xb, p) = \bigvee\{\mu(q, x, r) \wedge \mu(r, b, p) : r \in Q\}.$$

$\mathcal{M}$ is called switching if and only if $\mu(q, a, p) = \mu(p, a, q)$ and $\mathcal{M}$ is called commutative if and only if $\mu(q, ab, p) = \mu(q, ba, p)$ for all $q, p \in Q$ and $a, b \in X$. If $\mathcal{M}$ is switching and commutative, then $\mathcal{M}$ is called fuzzy finite switchboard automata (Sato and Kuroki, 2002).

Definition 3.1: (Qiu, 2002)

Let $\mathcal{L}$ be a complete residuated lattices and X be an (finite) alphabet. A fuzzy automaton over $\mathcal{L}$ and X, or simply a fuzzy automaton is a quadruple $\mathcal{M} = (Q, \delta, \sigma, \tau)$, where

a) Q is a non-empty set, called the finite set of states,

b) $\delta: Q \times X \times Q \to L$ is a fuzzy subset of $Q \times X \times Q$, called the fuzzy transition function,

c) $\sigma: Q \to L$ is a fuzzy subsets of Q, called the fuzzy set of input symbol,

d) $\tau: Q \to L$ is a fuzzy subsets of Q, called the fuzzy set of terminal states.

$\delta(q, x, p)$ can be interpreted as the degree to which an input letter $x \in X$ causes a transition from a state $q \in Q$ into a state $p \in Q$. Meanwhile, $\sigma(q)$ and $\tau(q)$ can be interpreted as the degrees to which q is respectively an input state and a terminal state. Assume that the input alphabet X is finite, but from methodological reasons the set of states Q is allowed to be infinite. A fuzzy automaton whose set of states is finite is called a finite fuzzy automaton.

Definition 3.2: (Qiu, 2002)

Let X^*denote the free monoid over the alphabet X. The mapping δ can be extended up to a mapping $\delta^*: Q \times X^* \times Q \to L$:

If $q, p \in Q$ and $e \in X^*$ is the empty word, then

$$\delta^*(q, e, p) = \begin{cases} 1 & q = p \\ 0 & otherwise' \end{cases}$$

and if $q, p \in Q, u \in X^*$ and $x \in X$, then

$$\delta^*(q, ux, p) = \bigvee_{r \in Q} (\delta^*(q, u, r) \otimes \delta(r, x, p)).$$

We have that for all $q, p \in Q$ and $u, v \in X^*$,

$$\delta^*(q, uv, p) = \vee_{r \in Q} (\delta^*(q, u, r) \otimes \delta(r, v, p)) \qquad (3.1)$$

In a way that if for any $u \in X^*$ we define a fuzzy relation δ_u on Q by $\delta_u(q,p) = \delta_*(q,u,b), \forall q,p \in Q$ called the fuzzy transition relation determined by u, then equation (3.1) can be written as

$$\delta_{uv} = \delta_u \circ \delta_v \quad u,v \in X^*. \tag{3.2}$$

Definition 3.3: (Holcombe, 1982)

The switching fuzzy automaton $\mathcal{M}_2 = (Q_2, \delta_2, \sigma_2, \tau_2)$ of a fuzzy automaton $\mathcal{M}_1 = (Q_1, \delta_1, \sigma_1, \tau_1)$whose fuzzy transition function and fuzzy sets of initial and terminal states are defined by

$$\delta_2(q_2, x, p_2) = \delta_1(p_1, x, q_1)$$

for all $q_1, p_1 \in Q_1$, $q_2, p_2 \in Q_2$ and $x \in X$, $\sigma_2 = \tau_1$ and $\tau_2 = \sigma_1$. In other words,

$$\delta_2(x) = (\delta_1(x))^{-1}$$

for each $x \in X$.

Remark 3.0

Note that, the switching or reversing fuzzy automaton $\mathcal{M}_2$ is obtained from $\mathcal{M}_1$ by exchanging fuzzy sets of initial and final states and "reversing" all the transitions. Besides that, in fact the multiplication $\otimes$ is commutative.

3.3.1 Algorithm construction to check the validity of switchboard automata

The procedure below shows the algorithm construction through computer language to check the validity of switchboard automata.

Input: the set Q of (n states) states of FFSA $\mathcal{M} = (Q, \delta, \sigma, \tau)$;

the set X of inputs alphabets of $\mathcal{M}$;

the transition table δ of $\mathcal{M}$=

Output: YES, if $\mathcal{M}$ is FFSA with n states, or NO, if $\mathcal{M}$ is not FFSA.

Procedure:

Step 1: Enter the state transition $\delta_{x_1}, \delta_{x_2}, \dots, \delta_{x_n}$.

Step 2: Set i be the initial value, $i = 1$ and $n \geq 2$.

Step 3: for $i \leq n - 1$, calculate $\delta_{x_i}\delta_{x_{i+1}}$ and $\delta_{x_{i+1}}\delta_{x_i}$

- If $\delta_{x_i}\delta_{x_{i+1}} \neq \delta_{x_{i+1}}\delta_{x_i}$ then STOP, the output $\mathcal{M}$ is not commutative;
- If $\delta_{x_i}\delta_{x_{i+2}} = \delta_{x_{i+2}}\delta_{x_i}$, recalculate $\delta_{x_i}\delta_{x_{i+2}}$and $\delta_{x_{i+2}}\delta_{x_i}$;
- If both are not equal then STOP, the output $\mathcal{M}$ is not commutative, NO;
- Otherwise, recalculate $\delta_{x_i}\delta_{x_{i+3}}$ and $\delta_{x_{i+3}}\delta_{x_i}$ and so on;
- If necessary, calculate until $\delta_{x_i}\delta_{x_n}$ and $\delta_{x_n}\delta_{x_i}$;
- If both are not equal, the output $\mathcal{M}$ is not commutative, NO;
- If both are equal Go to Step 4.

Step 4: $i = i + 1$ repeat Step 3.

Step 5: $i = n$, STOP, the output $\mathcal{M}$ is commutative, YES.

Step 6: for $i \leq n$, calculate $\delta_{x_i}(q,p)$and $\delta_{x_i}(p,q) \forall p, q \in Q$.

- If $\delta_{x_i}(q,p) \neq \delta_{x_i}(p,q)$, then STOP, the output $\mathcal{M}$ is not switching;
- If $\delta_{x_i}(q,p) \neq \delta_{x_i}(p,q)$, recalculate $\delta_{x_{i+1}}(q,p)$and $\delta_{x_{i+1}}(p,q)$;
- If both are not equal then STOP, the output $\mathcal{M}$ is not switching, NO;
- Otherwise, recalculate $\delta_{x_{i+2}}(q,p)$ and $\delta_{x_{i+2}}(p,q)$, and so on;
- If both are not equal, the output $\mathcal{M}$ is not switching, NO;
- If both are equal Go to Step 7.

Step 7: $i = i + 1$ repeat Step 6.

Step 8: $i = n$, STOP, the output $\mathcal{M}$ is switching, YES.

The algorithm of Fuzzy Finite Switchboard Automata is shown in Figure 3.1

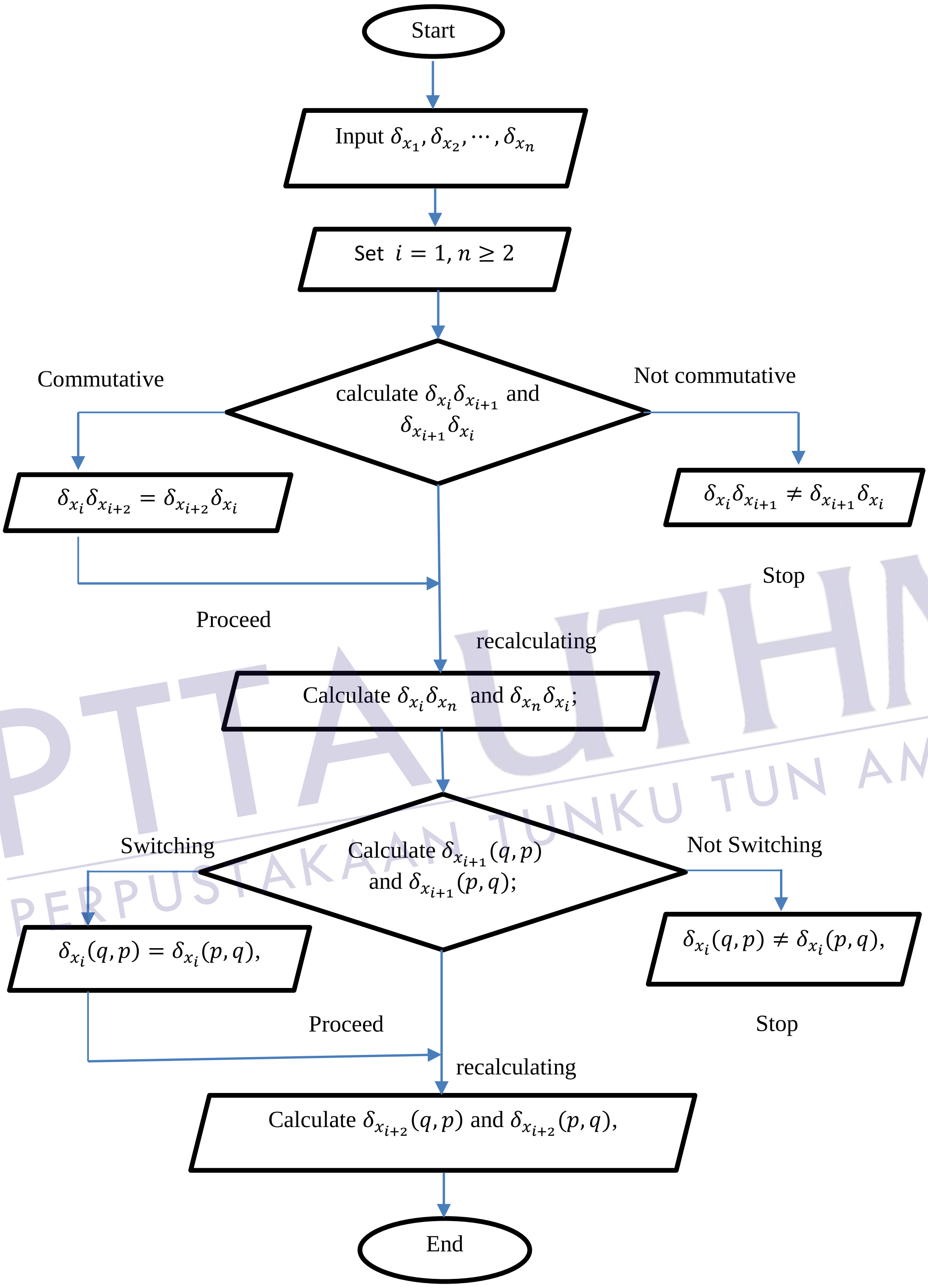

Figure 3.1: Flowchart for the algorithm construction of the Fuzzy Finite Switchboard Automata

3.3.2 General algorithm for Fuzzy Finite Switchboard Automata by using Complete Residuated Lattices

The new general algorithm for Fuzzy Finite Switchboard Automata (FFSA) by using CRL is given below:

1. Assume the systems follow by the properties of switchboard which are switching and commutative.
2. Enter the input in FFA and the output in FFSA.
3. State transition, δ_x and δ_y consists of unit interval [0,1] denoted as the grade of membership of state transition.
4. Define the next states and every state.
5. $q_0 \in Q$ the initial state is taken as input.
6. Find the membership value for each state by using CRL by entering CRL's equation, $min(1 - \delta_x + \delta_y, 1)$

   6.1 Suppose that $\delta(q, x, p)$ as a transition, where $p, q \in Q$ and $x \in X$. The values are given to each input symbols.

   6.2 By using CRL, substitute the value of δ_x and δ_y regarding the path that are chosen to obtain the membership value.

   6.3 Repeat the step 6.2 to obtain new membership value for another path.
7. Choose the best path or transition state based on the membership values, $\delta_n \geq 0.5$.

3.4 Subsystem

A subsystem is one of topological properties. A system is a group of parts that works together to perform a function. Meanwhile, a subsystem is an independent system that is an element of a larger system. It is smaller than the containing system that only provides some of the function that the larger system provides. A subsystem is needed in a machine because it can help manage the inherent complexity of the problem and implement according to the requirement. Besides, it also can identify and solve the problem simpler which focusing the certain partition or group only.

3.5 General Fuzzy Automata

General Fuzzy Automata (GFA, in short) was firstly proposed by Doostfatemeh and Kermer (2005). They have found the way to resolve the multi-membership and the problem to assign membership values to active states of a fuzzy automaton. By using the idea from Doorstfatemeh and Kermer, the concept of general fuzzy switchboard automaton (GFSA) is introduced and some efficient algebraic structure is proposed to study GFSA. Mainly, the algebraic approach to automata theory relies on semigroup where it is a generalization of the concept of a group. Since in a system consists enormous set of states, it is easier to explore the space of all possible finite computations by listing the semigroups. In chapter 6, certain basic definitions and results are presented and some elementary properties of transformation semigroup in GFSA are described and known as General Fuzzy Switchboard Transformation Semigroup (GFSTS).

Definition 3.4: (Doorstfatemeh & Kremer, 2005)
A general fuzzy automaton (GFA) is an eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ where

a) Q is a finite set of states, $Q = \{q_1, q_2, \cdots, q_n\}$,
b) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \cdots, a_m\}$,
c) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$,
d) Z is a finite set of output symbols, $Z = \{b_1, b_2, \cdots, b_k\}$,
e) $\omega: Q \to Z$ is the non-fuzzy output function,
f) $F_1: [0,1] \times [0,1] \to [0,1]$ is the membership assignment function,
g) $\tilde{\delta}: (Q \times [0,1]) \times \Sigma \times Q \xrightarrow{F_1(\mu,\delta)} [0,1]$ is the augmented transition function,
h) $F_2: [0,1]^* \to [0,1]$ is a multi-membership resolution function.

The function has two parameters μ and δ that represent as $F_1(\mu, \delta)$. μ is the membership value of a predecessor and δ is the weight of transition. Based on the definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}\left((q_i, \mu^t(q_i)), a_k, q_j\right) = F_1\left(\mu^t(q_i), \delta(q_i, a_k, q_j)\right). \quad (3.3)$$

It means that the membership value of the state q_j at time $t+1$ is figured by function F_1 using the weight of the transition, δ and the membership value of q_i at time t. Usually, the options for $F(\mu,\delta)$ are $\max\{\mu,\delta\}, \min\{\mu,\delta\}$ and $(\frac{\mu+\delta}{2})$. The multi-membership resolution function, F_2 resolves the multi-membership active state and assigns a single truth value to them.

Let $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \sum, \delta(q', a, q) \in \Delta\}, \forall i \geq 1$.

Since $Q_{act}(t_i)$ is a fuzzy set, then $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$. Hereafter, simply denote as $q \in (Q_{act}(t_i))$ and $T \subset (Q_{act}(t_i))$. The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j indicates the multi-membership resolution algorithm.

3.5.1 **Algorithm** (Doorstfatemeh & Kremer, 2005)

This algorithm is for multi-membership resolution. If there are various simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a united membership value to it:

i. Each transition weight $\tilde{\delta}(q_i, a_k, q_j)$ together with membership value of state $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value that is called as v_i,

$$v_i = \tilde{\delta}\left((q_i, \mu^t(q_i)), a_k, q_j\right) = F_1\left(\mu^t(q_i), \delta(q_i, a_k, q_j)\right). \qquad (3.4)$$

ii. These truth values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2.

iii. The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = F^{n}_{2_{i=1}}\left[F_1\left(\mu^t(q_i), \delta(q_i, a_k, q_j)\right)\right]. \qquad (3.5)$$

Where

n is the number of simultaneous transitions to the active state q_j at time $t+1$.

$\delta(q_i, a_k, q_j)$ is the weight of a transition from q_i to q_j with input a_k.

$\mu^t(q_i)$ is the membership value of q_i at time t.

$\mu^{t+1}(q_j)$ is the final membership value of q_j at time $t+1$.

Definition 3.5: (Horry, 2016)

Let $\tilde{F} = (Q, \Sigma, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a general fuzzy automaton, which is defined in Definition 3.1. Let define max-min bipolar general fuzzy automata of the form:

$$\tilde{F} = (Q, \Sigma, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$$

Where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \cdots\}$ and for every $i, i \geq 0$:

$$\tilde{\delta}^*\left((q, \mu^{t_i}(q)), \Lambda, p\right) = \begin{cases} 1, & q = p, \\ 0, & otherwise \end{cases}$$

And for every $i, i \geq 1$: $\tilde{\delta}^*\left((q, \mu^{t_{i-1}}(q)), \mathrm{u}_i, p\right) = \tilde{\delta}\left((q, \mu^{t_{i-1}}(q)), \mathrm{u}_i, p\right)$,

$$\tilde{\delta}^*\left((q, \mu^{t_{i-1}}(q)), \mathrm{u}_i u_{i+1}, p\right) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}\left((q, \mu^{t_{i-1}}(q)), \mathrm{u}_i, q'\right) \wedge \tilde{\delta}\left((q', \mu^{t_i}(q')), u_{i+1}, p\right))$$

And recursively

$$\tilde{\delta}^*\left((q, \mu^{t_0}(q)), \mathrm{u}_1 u_2 \cdots u_n, p\right) = \vee \tilde{\delta}\left((q, \mu^{t_0}(q)), \mathrm{u}_1, p_1\right) \wedge \tilde{\delta}\left((p_1, \mu^{t_1}(p_1)), u_2, p_2\right)) \wedge \cdots \wedge$$

$$\tilde{\delta}\left((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p\right) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1})\},$$

In which $u_i \in \Sigma, \forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be $u_i, \forall 1 \leq i \leq n-1$.

Definition 3.6: (Moderson & Malik, 2002)

Let $(X,*)$ be a semigroup with identity λ and $\equiv$ be an equivalence relation on X. Then $\equiv$ is called a right (left) congruence relation on X if $\forall x, y, z \in X, x \equiv y \leftrightarrow x * z \equiv y * z (z * x \equiv z * y)$.

3.5.2 Algorithm for construction of the General Fuzzy Switchboard Automata

The new simple algorithm which generates the switchboard in general fuzzy automata is provided. Let $\tilde{F} = (Q, \Sigma, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a max-min bipolar general fuzzy automaton. When the number of input elements is 1, then $\tilde{F}^*$ is switching and commutative. Next, set $\Sigma = \{a_1, a_2, \cdots, a_n\}$.

Input: the set Q of (n states) states of GFSA, $\tilde{F}^* = \left(Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2\right)$;

the set Σ of input alphabets of $\tilde{F}^*$;

the augmented transition table $\tilde{\delta}^*$ of $\tilde{F}^*$.

Output: YES, if $\tilde{F}^*$ is GFSA with n states, or NO, if $\tilde{F}^*$ is not GFSA.

Procedure:

Step 1: Enter the state transition $\tilde{\delta}^*_{a_1}, \tilde{\delta}^*_{a_2}, \cdots, \tilde{\delta}^*_{a_n}$.

Step 2: Set i be the initial value, $i = 1$ and $n \geq 2$.

Step 3: for $i \leq n - 1$,

- Calculate $\tilde{\delta}^*_{a_i}\tilde{\delta}^*_{a_{i+1}}((q, \mu^{t_{i-1}}(q)), p)$ and $\tilde{\delta}^*_{a_{i+1}}\tilde{\delta}^*_{a_i}\left((q, \mu^{t_{i-1}}(q)), p\right) \forall q, p \in Q$
- If $\tilde{\delta}^*_{a_i}\tilde{\delta}^*_{a_{i+1}}((q, \mu^{t_{i-1}}(q)), p) \neq \tilde{\delta}^*_{a_{i+1}}\tilde{\delta}^*_{a_i}\left((q, \mu^{t_{i-1}}(q)), p\right)$ then $STOP$, the output $\tilde{F}^*$ is not commutative;
- If $\tilde{\delta}^*_{a_i}\tilde{\delta}^*_{a_{i+1}}\left((q, \mu^{t_{i-1}}(q)), p\right) = \tilde{\delta}^*_{a_{i+1}}\tilde{\delta}^*_{a_i}\left((q, \mu^{t_{i-1}}(q)), p\right)$, recalculate $\tilde{\delta}^*_{a_i}\tilde{\delta}^*_{a_{i+2}}\left((q, \mu^{t_{i-1}}(q)), p\right)$ and $\tilde{\delta}^*_{a_{i+2}}\tilde{\delta}^*_{a_i}\left((q, \mu^{t_{i-1}}(q)), p\right)$;
- If both are not equal then $STOP$, the output $\tilde{F}^*$ is not commutative, NO;
- Otherwise recalculate $\tilde{\delta}^*_{a_i}\tilde{\delta}^*_{a_{i+3}}\left((q, \mu^{t_{i-1}}(q)), p\right)$ and $\tilde{\delta}^*_{a_{i+3}}\tilde{\delta}^*_{a_i}\left((q, \mu^{t_{i-1}}(q)), p\right)$ and so on;
- If necessary, calculate until $\tilde{\delta}^*_{a_i}\tilde{\delta}^*_{a_n}\left((q, \mu^{t_{i-1}}(q)), p\right)$ and $\tilde{\delta}^*_{a_n}\tilde{\delta}^*_{a_i}\left((q, \mu^{t_{i-1}}(q)), p\right)$;
- If both are not equal, the output $\tilde{F}^*$ is not commutative, NO;
- If both are equal, Go to Step 4.

Step 4: $i = i + 1$ repeat Step 3.

Step 5: $i = n$, $STOP$, the output $\tilde{F}^*$ is commutative, YES.

Step 6: for $i \leq n$, calculate $\tilde{\delta}^*_{a_i}\left(\left(q, \mu^{t_i}(q)\right), p\right)$ and $\tilde{\delta}^*_{a_i}\left(\left(p, \mu^{t_i}(p)\right), q\right)$, $\forall q, p \in Q$.

- If $\tilde{\delta}^*_{a_i}\left(\left(q, \mu^{t_i}(q)\right), p\right) \neq \tilde{\delta}^*_{a_i}\left(\left(p, \mu^{t_i}(p)\right), q\right)$, then $STOP$, the output $\tilde{F}^*$ is not switching;
- If $\tilde{\delta}^*_{a_i}\left(\left(q, \mu^{t_i}(q)\right), p\right) = \tilde{\delta}^*_{a_i}\left(\left(p, \mu^{t_i}(p)\right), q\right)$, recalculate $\tilde{\delta}^*_{a_{i+1}}\left(\left(q, \mu^{t_i}(q)\right), p\right)$ and $\tilde{\delta}^*_{a_{i+1}}\left(\left(p, \mu^{t_i}(p)\right), q\right)$;
- If both are not equal, then $STOP$, the output $\tilde{F}^*$ is not switching, NO;
- Otherwise recalculate $\tilde{\delta}^*_{a_{i+2}}\left(\left(q, \mu^{t_i}(q)\right), p\right)$and $\tilde{\delta}^*_{a_{i+2}}\left(\left(p, \mu^{t_i}(p)\right), q\right)$and so on;
- If both are not equal, the output $\tilde{F}^*$ is not switching, NO;
- If both are equal, Go to Step 7.

Step 7: $i = i + 1$, repeat Step 6.

Step 8: $i = n$, $STOP$, the output is switching, YES.

3.6 Summary

The algorithm for the FFSA by using CRL has been presented and the examples are provided in Chapter 4. Generally, there are some problems to define the membership value for the active state in the machine if an active state has multi-membership value. Therefore, general fuzzy automata are used to resolve such multi-membership and the problem related to assigning membership value to active states of fuzzy automaton. In addition, the concept of switchboard is introduced in general fuzzy automaton and some efficient algebraic techniques are proposed in chapter 5. After the properties and definition of General Fuzzy Switchboard Automata (GFSA) are discussed, it is necessary to study the semigroup since most of the algebraic approaches for automata theory rely on semigroup. Some of the basic definitions and results of General Fuzzy Switchboard Transformation Semigroup (GFSTS) have been presented and elementary properties of semigroup are described in chapter 6.

CHAPTER 4

ALGEBRAIC PROPERTIES OF FUZZY FINITE SWITCHBOARD AUTOMATA

4.1 Introduction

This chapter discusses the uses of algebraic techniques to learn the structure of switchboard automata. For instance, Holcombe (1982) has introduced the algebraic technique of automata theory. However, the algebraic approaches on the theory of finite switchboard automata are still lacking, thus it is necessary to study the algebraic properties of a finite switchboard automata. In addition, some examples in real life are provided in this chapter regarding the algebraic properties of the finite switchboard automata. Moreover, the theory of fuzzy finite switchboard automata (FFSA) is introduced by utilizing the idea of general algebraic structure, such as complete residuated lattices (CRL). The role of CRL is to obtain enhanced membership grades. The notion of homomorphism, strong homomorphism and reverse homomorphism are established and some of its properties are shown. The subsystem of FFSA is studied and the set of switchboard subsystem-forms complete L-sublattices are shown. The algorithm of FFSA with CRL is given in Chapter 3 and an example is provided.

4.2 Algebraic properties

The two operations of addition and multiplication are linear transformation. Another operation of the basic algebraic properties is the dot product.

Table 4.1 Algebraic properties consist of two operations

Name	Properties	Explanations
Associative	**Additive** $A + (B + C) = (A + B) + C$ **Multiplicative** $A \cdot (B \cdot C) = (A \cdot B) \cdot C$	Grouping (parenthesis) more than two numbers and it works with addition and multiplication
Commutative	**Additive** $A + B = B + A$ **Multiplicative** $A \cdot B = B \cdot A$	Changing the order but it does not change the value.
Distributive	**Left distributive law** $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$ **Right distributive law** $(A + B) \cdot C = (A \cdot C) + (B \cdot C)$	Can be done two way which are left and right distributive law
Identity	**Additive identity** $A + 0 = A$ **Multiplicative identity** $A \cdot 1 = 1 \cdot A = A$	Additive identity is 0 and multiplicative identity is 1.
Inverse	**Additive inverse** $A + (-A) = 0$ **Multiplicative inverse** $A \cdot \frac{1}{A} = 1$ or $\frac{A}{B} \cdot \frac{B}{A} = 1$	The additive inverse of A is $-A$ while the multiplicative inverse of A non zero real number are $A \cdot \frac{1}{A} = 1$ or $\frac{A}{B} \cdot \frac{B}{A} =$ 1(reciprocals)

The table above represent the algebraic properties that consist of two operations which are addition and multiplication. The algebraic properties such as associative, commutative and identity will be used in chapter 4, 5 and 6.

4.3 Properties of switchboard automata

Let M_1 and M_2 are finite switchboard state machine (Sato and Kuroki, 2002). Let $(p_1, p_2), (q_1, q_2), \in Q_1 \times Q_2$ and $b = uv \in X, r \in Q_1, s \in Q_2$. Since M_1 and M_2 are commutative, then

$$\delta\{((p_1, p_2), uv, (q_1, q_2))\} = \delta\{(p_1, uv, q_1) \wedge (p_2, uv, q_2)\}$$
$$= \delta\{(p_1, u, r) \wedge (r, v, q_1) \wedge (p_2, u, s) \wedge (s, v, q_2)\}$$
$$= \delta\{(p_1, vu, q_1) \wedge (p_2, vu, q_2)\}$$

Therefore, $M_1 \wedge M_2$ are commutative.

Suppose,

$$\delta\{(p_1, p_2), b, (q_1, q_2)\} = \delta\{(p_1, b, q_1) \wedge (p_2, b, q_2)\}$$

As M_1 and M_2 are switching,

$$\delta\{(p_1, b_1, q_1)\} = \delta\{(q_1, b_1, p_1)\}$$
$$\delta\{(p_2, b_2, q_2)\} = \delta\{(q_2, b_2, p_2)\}$$

Thus,

$$\begin{aligned}\delta\{(p_1, p_2), b, (q_1, q_2)\} &= \delta\{(p_1, b, q_1) \wedge (p_2, b, q_2)\} \\ &= \delta\{(q_1, b, p_1) \wedge (q_2, b, p_2)\} \\ &= \delta\{(q_1, q_2), b, (p_1, p_2)\}\end{aligned}$$

Hence, $M_1 \wedge M_2$ are switching. Therefore, $M_1 \wedge M_2$ is finite switchboard automata restricted direct product of M_1 and M_2.

4.4 Product of Fuzzy Finite Switchboard Automata

The concept of restricted cascade product of the Fuzzy Finite Switchboard Automata (FFSA) is examined.

Definition 4.0: (Kavikumar et al., 2019)

Let $\mathcal{M}_1 = (Q_1, \delta_1, \sigma_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, \delta_2, \sigma_2, \tau_2)$ be complete FFSA. Define the restricted cascade product $\mathcal{M}_1 \varpi \mathcal{M}_2 = (Q_1 \times Q_2, X_2, \delta^{\varpi})$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to mapping $\varpi: X_2 \rightarrow X_1$ as,

$$\delta^{\varpi}\big((p_1, p_2), \sigma_2, (q_1, q_2)\big) = \delta_1(p_1, \varpi(\sigma_2), q_1) \wedge \delta_2(q_2, \sigma_2, p_2),$$

Where $\delta^{\varpi}: (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \rightarrow [0,1], \forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $\sigma_2 \in X_2$.

Example 4.4.1:((Restricted cascade product)

Let $M_1 = (Q_1, \delta_1, \sigma_1, \tau_1)$ and $M_2 = (Q_2, \delta_2, \sigma_2, \tau_2)$ be FFA's, where $Q_1 = \{p_1, p_2\}, Q_2 = \{q_1, q_2\}, X_1 = \{\sigma_1, \tau_1, \rho\}$, $X_2 = \{\sigma_2, \tau_2\}$ and δ_1 and δ_2 are defined as follows:

$$\delta_1(p_1, \sigma_1, p_1) = 0.5 \qquad \delta_1(p_2, \rho, p_1) = 0.7$$
$$\delta_1(p_2, \sigma_1, p_1) = 0.2 \qquad \delta_2(q_1, \sigma_2, q_1) = 0.6$$
$$\delta_1(p_1, \tau_1, p_2) = 0.2 \qquad \delta_2(q_2, \sigma_2, q_1) = 0.2$$
$$\delta_1(p_2, \tau_1, p_2) = 0.6 \qquad \delta_2(q_1, \tau_2, q_2) = 0.5$$
$$\delta_1(p_1, \rho, p_2) = 0.4 \qquad \delta_2(q_2, \tau_2, q_1) = 0.35$$

Now the function $\varpi: X_2 \to X_1$ is defined as

$\varpi(\sigma_2) = \sigma, \varpi(\tau_2) = \tau.$

Next, define the partial function $\delta^{\varpi}: (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \to [0,1]$ as:

$$\delta^{\varpi}\big((p_1, q_1), \sigma_2, (p_1, q_1)\big) = \delta_1(p_1, \varpi(\sigma_2), p_1) \wedge \delta_2(q_1, \sigma_2, q_1) = 0.5$$

$$\delta^{\varpi}\big((p_1, q_2), \sigma_2, (p_1, q_1)\big) = \delta_1(p_1, \varpi(\sigma_2), p_1) \wedge \delta_2(q_2, \sigma_2, q_1) = 0.2$$

$$\delta^{\varpi}\big((p_2, q_1), \sigma_2, (p_1, q_1)\big) = \delta_1(p_2, \varpi(\sigma_2), p_1) \wedge \delta_2(q_1, \sigma_2, q_1) = 0.2$$

$$\delta^{\varpi}\big((p_2, q_2), \sigma_2, (p_1, q_1)\big) = \delta_1(p_2, \varpi(\sigma_2), p_1) \wedge \delta_2(q_2, \sigma_2, q_1) = 0.2$$

$$\delta^{\varpi}\big((p_1, q_1), \tau_2, (p_2, q_2)\big) = \delta_1(p_1, \varpi(\tau_2), p_2) \wedge \delta_2(q_1, \tau_2, q_2) = 0.2$$

$$\delta^{\varpi}\big((p_1, q_2), \tau_2, (p_2, q_1)\big) = \delta_1(p_1, \varpi(\tau_2), p_2) \wedge \delta_2(q_2, \tau_2, q_1) = 0.2$$

$$\delta^{\varpi}\big((p_2, q_1), \tau_2, (p_2, q_2)\big) = \delta_1(p_2, \varpi(\tau_2), p_2) \wedge \delta_2(q_1, \tau_2, q_2) = 0.5$$

$$\delta^{\varpi}\big((p_2, q_2), \tau_2, (p_2, q_1)\big) = \delta_1(p_2, \varpi(\tau_2), p_2) \wedge \delta_2(q_2, \tau_2, q_1) = 0.35$$

And δ^{ϖ} is 0 elsewhere. It follows that $M_1 \varpi M_2 \cong M_1 \omega M_2$ is restricted cascade product.

Proposition 4.0:

Let $M_1 = (Q_1, \delta_1, \sigma_1, \tau_1)$ and $M_2 = (Q_2, \delta_2, \sigma_2, \tau_2)$ be FFA's. Then there exists $\omega: Q_2 \times X_2 \to X_1\ \forall \varpi: X_2 \to X_1$ such that $M_1 \varpi M_2 \cong M_1 \omega M_2$.

Proof:

Let ω be defined by $\omega(p_2, \sigma_2) = \varpi(\alpha(p_2, \sigma_2) \in\ Q_2 \times X_2$ where $\alpha: Q_2 \times X_2 \to X_2$ is a projection mapping by the definition and it well-defined. Let ξ be an identity map on X and η be an identity map on $Q_1 \times Q_2$.

Then $\delta^{\varpi}(\eta(p_1, p_2), \sigma_2, (q_1, q_2)) = \delta^{\varpi}((p_1, p_2), \sigma_2, (q_1, q_2))$

$$= \delta_1(p_1, \varpi(\sigma_2), q_1) \wedge (p_2, \sigma_2, q_2))$$

$$= \delta_1(p_1, \omega(p_2, \sigma_2), q_1) \wedge (p_2, \sigma_2, q_2))$$

$$= \delta^{\omega}((p_1, p_2), \sigma_2, (q_1, q_2))$$

$$= \delta^{\omega}(\eta(p_1, p_2), \xi(\sigma_2), \eta(q_1, q_2))$$

Hence, $M_1 \varpi M_2 \cong M_1 \omega M_2$.

Proposition 4.1:

Let $M_1 = (Q_1, S_1, \rho_1)$ and $M_2 = (Q_2, S_2, \rho_2)$ be FFA's. Denote that $\varpi: S(\mathcal{M}_2) \rightarrow S(\mathcal{M}_1)$ be a semigroup homomorphism. Then $\mathcal{M}_1 \varpi \mathcal{M}_2$ is a FFSA if and only if both $\mathcal{M}_1$ and $\mathcal{M}_2$ are FFSA's.

Proof:

Assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are FFSA's. Since ϖ is homomorphism while $\mathcal{M}_1$ and $\mathcal{M}_2$ are commutative, therefore

$$\begin{aligned}
\rho^{\varpi}\big((p_1, p_2), \sigma_2\tau_2, (q_1, q_2)\big) &= \rho_1(p_1, \varpi(\sigma_2\tau_2), q_1) \wedge \rho_2(p_2, \sigma_2\tau_2, q_2) \\
&= \rho_1(p_1, \varpi(\sigma_2)\varpi(\tau_2), q_1) \wedge \rho_2(p_2, \sigma_2\tau_2, q_2) \\
&= \rho_1(p_1, \varpi(\tau_2)\varpi(\sigma_2), q_1) \wedge \rho_2(p_2, \tau_2\sigma_2, q_2) \\
&= \rho_1(p_1, \varpi(\tau_2\sigma_2), q_1) \wedge \rho_2(p_2, \tau_2\sigma_2, q_2) \\
&= \rho^{\varpi}\big((p_1, p_2), \tau_2\sigma_2, (q_1, q_2)\big)
\end{aligned}$$

Thus $\mathcal{M}_1 \varpi \mathcal{M}_2$ is commutative for all $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2, \sigma_2, \tau_2 \in X_2$.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are switching, then

$$\rho_1(p_1, \varpi(\sigma_2), q_1) = \rho_1(q_1, \varpi(\sigma_2), p_1)$$
$$\rho_2(p_2, \varpi(\sigma_2), q_2) = \rho_2(q_2, \varpi(\sigma_2), p_2)$$

Thus

$$\begin{aligned}
\rho^{\varpi}\big((p_1, p_2), \sigma_2, (q_1, q_2)\big) &= \rho_1(q_1, \varpi(\sigma_2), p_1) \wedge \rho_2(q_2, \sigma_2, p_2) \\
&= \rho^{\varpi}((q_1, q_2), \sigma_2, (p_1, p_2))
\end{aligned}$$

Therefore, $\mathcal{M}_1 \varpi \mathcal{M}_2$ is switching and assume that $\mathcal{M}_1 \varpi \mathcal{M}_2$ is a FFSA.

4.5 Illustrative examples of switchboard automata in real life

This section is done by discussing some real-life examples which are the simple system of Pac-man game and microwave. The examples provided are to check the system whether it follows the properties of a switchboard automata or not. It is easier to understand by referring to the properties of the switchboard automata in section 4.3.

4.5.1 Pac-man game

In Pac-man game, Pac-man should wander the maze by eating all the Pac-dots, meanwhile the ghosts will chase Pac-man to avoid it from completing that stage. The ghosts roam the maze, trying to kill Pac-man then Pac-man needs to flee to avoid from losing a life. Besides that, the advantage is given to Pac-man to counter-attack the ghosts by eating flashing dots near the corners. It provided Pac-man the temporary ability to eat the ghosts. When the enemies are eaten, their eyes remain and return to the base. To make it in a clear view, the simple figure is provided below. The example shown is for checking the switchboard properties in the system only and not consider for fuzzy (Gribkoff, 2013).

Let $M = (Q, \sum, F)$ where

Q is the nonempty states

$\sum$ is the alphabets used

F is the set of final states

$Q = \{1,2,3,4\}$, $\sum = \{\sigma, \tau\}$ is defined by Figure 4.1. Determine whether the example fulfilled the algebraic properties of finite switchboard automata.

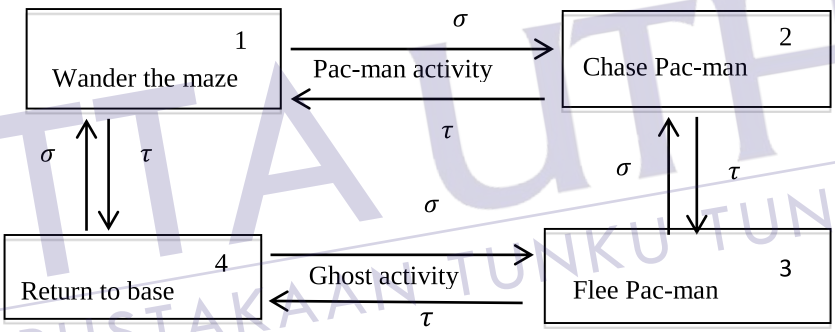

Figure 4.1: The simple system of Pac-man game

If the system fulfilled by the properties of switching and commutative, then the system is called a switchboard state machine. Firstly, the switching properties need to check in this system. Note that, if $pF_\sigma = q \Rightarrow qF_\sigma = p$ for each $p, q \in Q, \sigma \in \sum, \sigma \neq \emptyset$, then M is called switching. Here, based on the system of Figure 4.1, let's check the property of switching. From state 1 with the input symbol σ, the next state is state 2. If the state start with state 2 with same input symbol, τ the next state is state 1. Since the transition of input symbol from state 1 to state 2 and state 2 to state 1 are not the same such that $\delta(1, \sigma) = 2 \neq \delta(2, \tau) = 1$, then it has been shown that $pF_\sigma = q \neq qF_\sigma = p$.Therefore, the system of Example 4.5.1 is not switching. Note that, $\delta: Q \times \sum \rightarrow Q$ is a transition function.

Next, the commutative properties need to check from the same figure. If $qF_{\sigma\tau} = qF_{\tau\sigma}$ for each $q \in Q$, $\sigma, \tau \in \sum$, then M is called commutative.
By referring Figure 4.1, assume that

$$qF_{\sigma\tau} = \delta(1, \sigma, \tau) = 3$$
$$qF_{\tau\sigma} = \delta(1, \tau, \sigma) = 3$$

It has been shown that $qF_{\sigma\tau} = qF_{\tau\sigma}$. Therefore, the system of Pac-man game is a commutative. Conclusion can be made that Figure 4.1 is not a finite switchboard automaton because the system satisfied for one property only that is commutative. Therefore, the system of Pac-man game is a finite automaton.

4.5.2 Microwave

The one-minute microwave is another application of finite switchboard automata. It is a simple system with the following requirements (Ramnath and Dathan, 2003):

a) There is a single button available for the user.
b) If the door is closed and the button is pushed, the oven will be energized for one minute.
c) If the button is pushed while the oven is energized, the cooking time is increased by one minute.
d) If the door is open, pushing the button has no effect.
e) The oven has a light that is turned on when the door is open, and also when the oven is cooking. Otherwise the light is off.
f) Opening the door stops the cooking and clears the timer (i.e., remaining cooking time is set to zero).
g) When the cooking is complete (oven times out) a beeper is sounded and the light is turned off.

The Oven and the Timer are two subsystems where the Oven includes the Open Door state, Idle/Waiting state and Active state while the Timer includes Sleep state, Idle/Waiting state and Active state. These subsystems can be performed independently. However, these two subsystems need to communicate. Thus the switchboard is needed to maintain the communication between these separate components.

Let $\mathcal{M} = (Q, \sum, F)$, $Q = \{q_1, q_2, q_3, m_1, m_2, m_3\}$, $\sum = \{\sigma, \tau, on, off\}$ is defined by the action Figure 4.2.

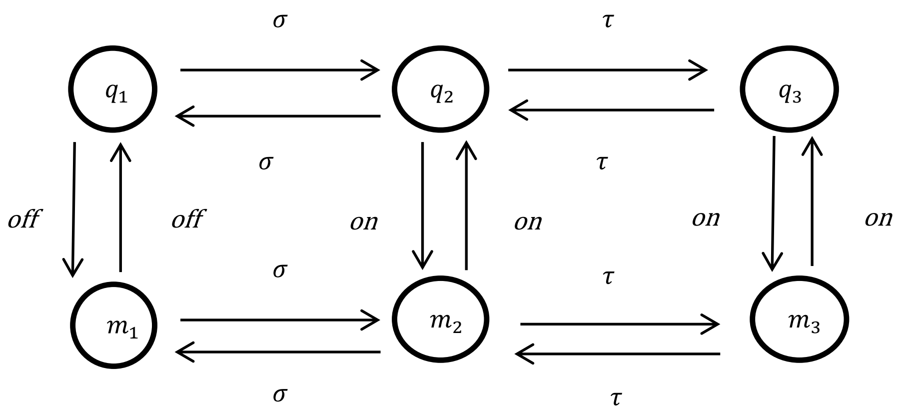

Figure 4.2: The state diagram of a microwave

Firstly, check whether the system of Figure 4.2 is switching.

If $pF_\rho = q \Rightarrow qF_\rho = p$ for each $p, q \in Q$, $\rho \in \sum$, $\rho \neq \emptyset$, then M is called switching. Here, from the system of microwave, let $pF_{off} = q$. From state q_1 to state m_1 with the input symbol, off is equal to the state which start from state m_1 to state q_1 with same input symbol. It has been shown that $pF_{off} = q \Rightarrow qF_{off} = p$. Therefore, the system of Figure 4.2 is switching.

Next, check for the commutative property. Let μ represents as a function.

$$\mu(m_2, on\tau, q_3) = \mu\{(m_2, on, q_2) \wedge (q_2, \tau, q_3)\}$$
$$\mu(m_2, \tau on, q_3) = \mu\{(m_2, \tau, m_3) \wedge (m_3, on, q_3)\}$$

From this equation, the system shows that it is commutative. This is because when the transition starts at state $m_2$and ends at state q_3, the input symbol is different, but the beginning state and finishing state are the same. Hence, this system is a commutative system. As a conclusion, Example 4.5.2 is a finite switchboard automaton since the system fulfilled switching and commutative properties.

4.6 Fuzzy Finite Switchboard Automata by Complete Residuated Lattices

Complete Residuated Lattice (CRL) is an algebraic structure that has a strong association with the mathematical model. The difference between complete lattice and normal lattice is normal lattices have to satisfy only for nonempty finite subsets, while for complete lattice, the bound condition has to be satisfied by all subsets (Jevanovic, 2005). However, all lattices with a finite number of elements are always complete lattices. The definition 3.1 of CRL is already mentioned in Chapter 3. In this section, CRL is applied to the fuzzy finite switchboard automata to enhance the membership value. New definitions and theorems are introduced by applying CRL and the proving is shown below.

Definition 4.1:

The definition of 3.1 and the properties of switchboard automata are combined to produce the definition of FFSA by CRL. Let $\mathcal{M} = (Q, \delta, \sigma, \tau)$ be a fuzzy automata over $\mathcal{L}$ and X.

1) $\mathcal{M}$ is called switching if and only if $\delta_x(q,p) = (\delta_x)(p,q)$ for $\forall q,p \in Q$ and $x \in X$.
2) $\mathcal{M}$ is called commutative if and only if $\delta_{xy}(p,q) = \delta_{yx}(p,q)$ for $\forall q,p \in Q$ and $x, y \in X$.

If $\mathcal{M}$ is switching and commutative, then $\mathcal{M}$ is called fuzzy finite switchboard automata over $\mathcal{L}$ and X.

Now, the proof of the following results for the next three theorems is straightforward.

Theorem 4.0:

The following conditions are equivalent:

i. $(L, \otimes, 1)$ is an $\mathcal{L}$-monoid, that is to say, the multiplication is distributive to finite joins, $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$, $(y \vee z) \otimes x = (y \otimes x) \vee (z \otimes x)$.

ii. For any $\mathcal{M}$ over $\mathcal{L}$ and X and for any $q, p \in Q$ and $x, y \in X^*, \delta^*_{xy}(q,p) = \delta^*_{yx}(q,p)$.

iii. For any $\mathcal{M}$ over $\mathcal{L}$ and X and for any $q, p \in Q$ and $x \in X^*$, $\delta^*_x(q,p) = \delta^*_x(p,q)$.

Proof:

i. The proof is straightforward.

ii. $\mathcal{M}$ is said to be commutative if it satisfied $\delta(q, xy, p) = \delta(q, yx, p)$. By induction on the length of y. If $|y| = 0$, then $y = \varepsilon$ and

$$\delta^*(q,xy,p) = \delta^*(q,x\varepsilon,p) = \delta^*(q,x,p) = \delta^*(q,x,p)\cdot\varepsilon = \delta^*(q,x,p)\cdot\delta^*(q,y,p)$$

$$= \vee_{r\in Q}\big(\delta^*(q,x,r)\cdot\delta^*(r,\varepsilon,p)\big)$$

$$= \delta^*(q,x,p)$$

$$= \vee_{r\in Q}\big(\delta^*(q,\varepsilon,r)\cdot\delta^*(r,x,p)\big)$$

$$= \vee_{r\in Q}\big(\delta^*(q,y,r)\cdot\delta^*(r,x,p)\big)$$

$$= \delta^*(q,yx,p) \text{ (shown)}$$

iii. $\mathcal{M}$ is said to be switching if it satisfied $\delta(q,x,p) = \delta(p,x,q)$. Since $p,q \in Q$ and $x \in X^*$. The result was proven by induction on $|x| = n$. Assume that $x = \varepsilon$ while $n = 0$. Then $\delta^*(q,x,p) = \delta^*(q,\varepsilon,p) = \delta^*(p,\varepsilon,q) = \delta^*(p,x,q)$. Thus, the theorem holds for $x = \varepsilon$.

Next, assume that the results hold $\forall u \in X^*$ such that $|u| = n-1$ and $n > 0$. Let $t \in X$ and $x \in X^*$ be such that $x = ut$. Then

$$\delta^*(q,x,p) = \delta^*(q,ut,p)$$

$$= \vee_{r\in Q}(\delta(q,u,r)\cdot\delta(r,t,p))$$

$$= \vee_{r\in Q}(\delta\big((r,u,q)\cdot\delta(p,t,r)\big))$$

$$= \vee_{r\in Q}\delta\left((p,t,r)\cdot\delta\big((r,u,q)\big)\right)$$

$$= \delta^*(p,tu,q)$$

$$= \delta^*(p,ut,q)$$

$$= \delta^*(p,x,q)$$

Hence, the result is true for $|u| = n$. The proof is completed.

Definition 4.2:

Let $\mathcal{M}_1 = (Q_1,\delta_1,\sigma_1,\tau_1)$ and $\mathcal{M}_2 = (Q_2,\delta_2,\sigma_2,\tau_2)$ be fuzzy finite automata over $\mathcal{L}$ and X. A (strong) homomorphism from $\mathcal{M}_1$ and $\mathcal{M}_2$ is a pair (α,β) of mappings

$\alpha: \mathcal{M}_1 \to \mathcal{M}_2$ and $\beta: X_1 \to X_2$, such that $\delta_1(q,x,p)(=_{\mathcal{L}}) \leq_{\mathcal{L}} \delta_2(\alpha(q),\beta(x),\alpha(p))$for any $q,p \in Q_1$ and $x \in X_1$.

Theorem 4.1:

Let $\mathcal{M}_1 = (Q_1,\delta_1,\sigma_1,\tau_1)$ be a commutative fuzzy finite automata over $\mathcal{L}$ and X, and $\mathcal{M}_2 = (Q_2,\delta_2,\sigma_2,\tau_2)$ be a fuzzy finite automata over $\mathcal{L}$ and X. Let $(\alpha,\beta): \mathcal{M}_1 \to \mathcal{M}_2$ be an onto strong homomorphism. Then $\mathcal{M}_2$ is commutative fuzzy finite automata over $\mathcal{L}$ and X.

Proof:

Let $q_1,p_1 \in Q_1$ and $q_2,p_2 \in Q_2$. Denote $\alpha(q_1) = q_2$ and $\alpha(p_1) = p_2$. Let $x_2,y_2 \in X_2$ and there exists $x_1,y_1 \in X_1$ such that $\beta(x_1) = x_2$ and $\beta(y_1) = y_2$ since $\mathcal{M}_1$ is commutative, then

$$\begin{aligned}
\delta_2(q_2,x_2y_2,p_2) &= \delta_2(\alpha(q_1),\beta(x_1)\beta(y_1),\alpha(p_1)) \\
&= \delta_2(\alpha(q_1),\beta(x_1y_1),\alpha(p_1)) \\
&= \vee\{\delta_1(q_1,x_1y_1,r_1) | r_1 \in \mathcal{M}_1, \alpha(r_1) = \alpha(p_1)\} \\
&= \vee\{\delta_1(q_1,y_1x_1,r_1) | r_1 \in \mathcal{M}_1, \alpha(r_1) = \alpha(p_1)\} \\
&= \delta_2(\alpha(q_1),\beta(y_1x_1),\alpha(p_1)) \\
&= \delta_2(q_2,y_2x_2,p_2)
\end{aligned}$$

Definition 4.3:

Let $\mathcal{M}_1 = (Q_1,\delta_1,\sigma_1,\tau_1)$ and $\mathcal{M}_2 = (Q_2,\delta_2,\sigma_2,\tau_2)$ be fuzzy finite automata over $\mathcal{L}$ and X. A reverse homomorphism from $\mathcal{M}_1$ and $\mathcal{M}_2$ is a pair (α,β) of mappings $\alpha: \mathcal{M}_1 \to \mathcal{M}_2$ and $\beta: X_1 \to X_2$, such that

$$\delta_2\big(\alpha(q),\beta(x),\alpha(p)\big) = \vee\{\delta_1(s,x,t) | s,t \in Q_1, \alpha(t) = \alpha(q), \alpha(s) = \alpha(p)\}$$

for any $q,p \in Q_1$ and $x \in X_1$.

Theorem 4.2:

Let $\mathcal{M}_1 = (Q_1,\delta_1,\sigma_1,\tau_1)$ be a fuzzy finite switchboard automata over $\mathcal{L}$ and X, and $\mathcal{M}_2 = (Q_2,\delta_2,\sigma_2,\tau_2)$ be a fuzzy finite automata over $\mathcal{L}$ and X. Let $(\alpha,\beta): \mathcal{M}_1 \to \mathcal{M}_2$

be an onto reverse homomorphism. Then $\mathcal{M}_2$ is fuzzy finite switchboard automata over $\mathcal{L}$ and X.

Proof:

Let $q_1, p_1 \in Q_1$ and $q_2, p_2 \in Q_2$. Denote $\alpha(q) = s$ and $\alpha(p) = t$. Let $x_1, x_2 \in X$ where $\beta(x) = x$. Assume that the results hold $\forall u \in X^*$ where $|u| = n - 1$ and $n > 0$. Let $v \in X$ and $x \in X^*$ such that $x = uv$. Then

$$\delta_2\big(\alpha(q), \beta(x), \alpha(p)\big) = \vee\{\delta_1(s, x, t) | s, t \in Q_1, \alpha(t) = \alpha(q), \alpha(s) = \alpha(p)\}$$

$$= \vee\{\delta_1(s, uv, t) |\ s, t \in Q_1, uv \in x, \alpha(t) = \alpha(q), \alpha(s) = \alpha(p)\ \}$$

$$= \vee_{r \in Q}(\delta_1(s, u, r) \cdot \delta_1(r, v, t))$$

$$= \vee_{r \in Q}(\delta_1(r, u, s) \cdot \delta_1(t, v, r))$$

$$= \vee_{r \in Q}(\delta_1(t, v, r) \cdot \delta_1(r, u, s))$$

$$= (\delta_1(t, vu, s)$$

$$= (\delta_1(t, uv, s)$$

$$= \vee\{\delta_1(t, x, s) | s, t \in Q_1, \alpha(t) = \alpha(q), \alpha(s) = \alpha(p)\}$$

$$= \delta_1\big(\alpha(p), \beta(x), \alpha(q)\big)$$

4.9 Fuzzy finite switchboard subsystem

Let $\mathcal{S} = (S, \oplus, \otimes, 0, 1)$ be a semiring with the zero 0 and the identity 1. Let $\mathcal{M}(Q) = (L^Q, \wedge, \vee, \emptyset, Q)$ is a complete lattice with the least element $\emptyset$ and the greatest element Q. For any $\lambda \in L$ and $m \in L^Q$, let us define the left scalar multiplication λm and the right scalar multiplication $m\lambda$ as follows: $\lambda m(q) = \lambda \otimes m(q)$ and $m\lambda(q) = m(q) \otimes \lambda$, for every $q \in Q$. Due to the commutativity of the multiplication $\otimes$, the left and right scalar multiplications coincide, for instance $\lambda m = m\lambda$, for all $q \in Q$. The lattice $\mathcal{M}(Q)$ equipped with this scalar multiplication will be denoted by $\mathcal{M}_{\otimes}(Q)$ and called the $\mathcal{L}$-lattice of fuzzy subsets of the set Q. Any subset of L^Q which is closed under scalar multiplication and arbitrary meets and joins, and it contains the least and the greatest element of $\mathcal{M}(Q)$ will be called a complete $\mathcal{L}$-sublattice of $\mathcal{M}_{\otimes}(Q)$ (Ignjatovic et al., 2013). In this section, the notion of switchboard subsystem of fuzzy finite switchboard automata forms a complete $\mathcal{L}$-sublattice is introduced.

Definition 4.4:
Let over $\mathcal{M} = (Q, \delta, \sigma, \tau)$ be a fuzzy finite automata over $\mathcal{L}$ and X. Let μ be a fuzzy subset of Q. Then μ is a fuzzy finite switchboard subsystem of $\mathcal{M}$, if

$$\mu(q) \otimes \delta_x(q, p) \leq \mu(p)$$

and

$$\mu(q) \otimes \delta_{xy}(q, p) \leq \mu(p)$$

for all $p, q \in Q$ and $x, y \in X$.

Theorem 4.3: (Ignjatovic et al., 2013)
The collection $\mathcal{S}(\mathcal{M})$ of all fuzzy finite switchboard subsystems of a fuzzy finite switchboard automata $\mathcal{M}$ forms a complete $\mathcal{L}$-sublattice of $\mathcal{M}_{\otimes}(Q)$.

Proof:
It is very clear that the well-defined collection of $\mathcal{S}(\mathcal{M})$ of all fuzzy finite switchboard subsystems can be satisfied both reverse and commutative by definition 4.1 and 4.4 and theorem 4.0. Moreover, it is easy to check that the set of all fuzzy finite switchboard subsystems of fuzzy finite switchboard automata is closed under arbitrary meets, joins and $\emptyset$ and Q belong to $\mathcal{S}(\mathcal{M})$. Thus $\mathcal{S}(\mathcal{M})$ is a complete $\mathcal{L}$-sublattice of $\mathcal{M}_{\otimes}(Q)$.

4.8 Example of Fuzzy Finite Switchboard Automata by using Complete Residuated Lattices

Consider a FFSA system over $\mathcal{L}$ and X as given below. Let $\mathcal{M} = (Q, \delta, \sigma, \tau)$ where

a) Q is a non-empty set, called the finite set of states,

b) $\delta: Q \times X \times Q \rightarrow L$ is a fuzzy subset of $Q \times X \times Q$, called the fuzzy transition function,

c) $\sigma: Q \rightarrow L$ is a fuzzy subsets of Q, called the fuzzy set of input symbol,

d) $\tau: Q \rightarrow L$ is a fuzzy subsets of Q, called the fuzzy set of terminal states.

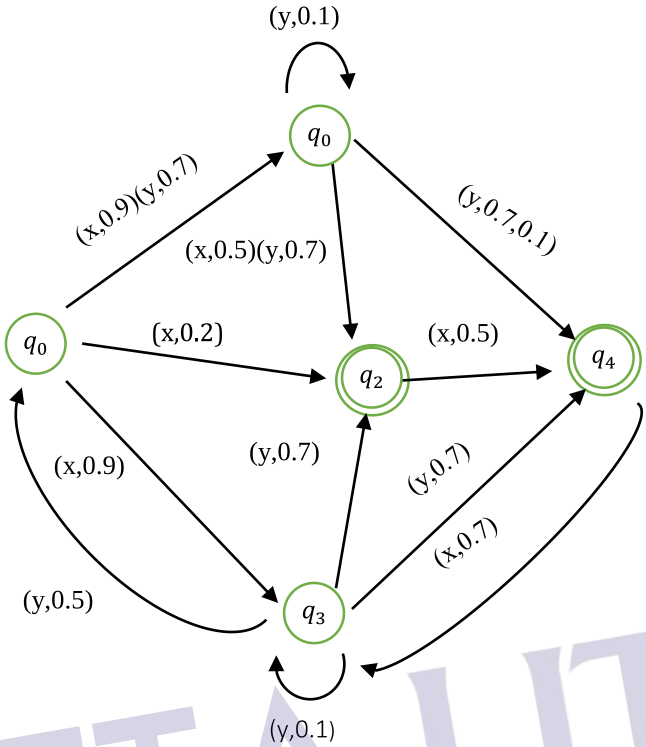

Figure 4.3: The system of Fuzzy Finite Switchboard Automata (FFSA)

Based on Figure 4.3, $q_0, q_1, q_2, q_3, q_4 \in Q$, and $x, y \in X$. It consists the membership values in an arbitrary set with two distinguished elements which values in $\mathcal{L}$. Let $q \xrightarrow{x} p$ is a transition where $q, p \in Q$ and $x \in X$. According to Figure 4.3, there are many possible paths to pass through from one state to another state. For instance, from q_0 to q_2 there are 2 possible paths which are $q_0 \rightarrow q_1 \rightarrow q_2$ or $q_0 \rightarrow q_3 \rightarrow q_2$ written as $\delta(q_0, xy, q_2)$and the input symbols are the same for both paths that are x and y respectively. Other example is from q_0 to q_4 where there are many possibilities of paths but in different membership values which are $q_0 \rightarrow^x q_1 \rightarrow^y q_4$ or $q_0 \rightarrow^x q_3 \rightarrow^y q_4$. Besides that, it can also be applied to the other states. The system is switching if $\delta_x(q, p) = (\delta_x)(p, q)$ for $\forall q, p \in Q$ and $x \in X$.

From Figure 4.3, $\delta_{xy}(q_0, q_4) = \delta_{xy}\left(q_{4,} q_0\right)$, therefore Figure 4.3 is switching. If $\delta_{xy}(p, q) = \delta_{yx}(p, q)$ for $\forall q, p \in Q$ and $x, y \in X$. Then $\mathcal{M}$ is called commutative. By referring to Figure 4.3, assume that $\delta_{xy}(q_0, q_2) = \delta_{yx}(q_0, q_2)$, thus Figure 4.3 is commutative. As a conclusion, Figure 4.3 is a finite switchboard automata.

4.8.1 Illustration of path calculation by using Complete Residuated Lattices

The equation of CRL is given by $x \rightarrow_L y = \min(1 - x + y, 1)$.

Table 4.2: The path calculation by using CRL

	Path
$\delta_{xy}(q_0) = \min(1 - 0.9 + 0.7, 1)$ $q_2 = 0.8$	$q_0 \rightarrow^x q_1 \rightarrow^y q_2$ or $q_0 \rightarrow^x q_3 \rightarrow^y q_2$
$\delta_{x}(q_0) = \min(1 - 0.2 + 0, 1)$ $q_2 = 0.8$	$q_0 \rightarrow^x q_2$
$\delta_{xy}(q_0) = \min(1 - 0.9 + 0.1, 1)$ $q_4 = 0.2$	$q_0 \rightarrow^x q_1 \rightarrow^y q_4$ or $q_0 \rightarrow^x q_3 \rightarrow^y q_4$
$\delta_{xy}(q_0) = \min(1 - 0.9 + 0.7, 1)$ $q_4 = 0.8$	$q_0 \rightarrow^x q_1 \rightarrow^y q_4$ or $q_0 \rightarrow^x q_3 \rightarrow^y q_4$
$\delta_{xx}(q_0) = \min(1 - 0.2 + 0.5, 1)$ $q_4 = 1$	$q_0 \rightarrow^x q_2 \rightarrow^x q_4$
$\delta_{x}(q_0) = \min(1 - 0.9 + 0, 1)$ $q_3 = 0.1$	$q_0 \rightarrow^x q_3$
$\delta_{x}(q_0) = \min(1 - 0.9 + 0, 1)$ $q_1 = 0.1$	$q_0 \rightarrow^x q_1$
$\delta_{xy}(q_0) = \min(1 - 0.9 + 0.1, 1)$ $q_1 = 0.2$	$q_0 \rightarrow^x q_1 \rightarrow^y q_1$
$\delta_{xy}(q_0) = \min(1 - 0.9 + 0.1, 1)$ $q_3 = 0.2$	$q_0 \rightarrow^x q_3 \rightarrow^y q_3$

Table 4.2 shows the calculation of paths by using CRL. The values that are obtained by using CRL represent the probability of the paths that might be chosen. For example, in Figure 4.3 there are many possibilities of the paths from q_0 to q_4. Therefore, the values that are nearest to 1 is assumed as the best path. Regarding to the Figure 4.3, from state q_0 to q_4, when the input symbol is x,$q_0 \xrightarrow{x} q_2 \xrightarrow{x} q_4$ the truth value is 1, that shows it is the best choice of path. This figure below is the selected path according to the membership values.

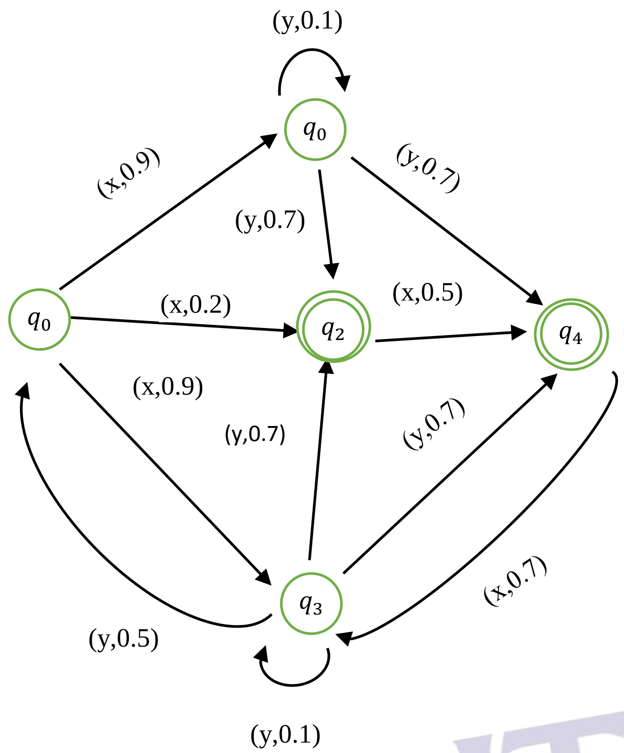

Figure 4.4: The simplest system after considering the membership value by CRL

4.9 Summary

This chapter studied some of the algebraic properties of FFSA, such that the restricted cascade product and semigroup homomorphism. Real-life applications are illustrated in section 4.5. In section 4.6, the CRL is applied in FFSA in order to enhance the membership value from [0,1] to more general algebraic structure. CRL is used as the structure of membership (truth) values. The basic algebraic properties of FFSA by CRL also established theorem 4.1, theorem 4.2 and theorem 4.3. Moreover, the subsystem of the FFSA is investigated in section 4.7. Since any subset of L^Q is closed under scalar multiplication and arbitrary meets and joins and also Q belongs to $\mathcal{S}(\mathcal{M})$, thus complete $\mathcal{L}$-sublattice of $\mathcal{M}_{\otimes}(Q)$ is introduced.

CHAPTER 5

GENERAL FUZZY SWITCHBOARD AUTOMATA

5.1 Introduction

The idea of the GFA has been introduced and studied by Doostfatemeh and Kremer (2005). They addressed the problem of the active state of fuzzy automata when the states have multi-membership values. Since then, many researchers continue and further the research regarding the GFA. Inspired by the idea of Doostfatemeh and Kremer (2005), this chapter studies the GFSA and its properties to resolve the multi-membership values. The applications regarding the General Fuzzy Switchboard Automata are also provided.

5.2 General Fuzzy Switchboard Automata

In this section, the definition and the notion of General Fuzzy Switchboard Automata (GFSA) are introduced. Some examples are provided regarding to the GFSA.

Definition 5.0:

The definition of 3.4 and the properties of switchboard automata are combined to produce the definition of GFSA.

Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a max-min general fuzzy automaton. Then

i. If $\tilde{F}^*$ is switching, it satisfies $\forall p, q \in Q, x \in \sum$ and let $i \geq 0$

$$\tilde{\delta}^*\left(\left(q, \mu^{t_i}(q)\right), x, p\right) = \tilde{\delta}^*\left(\left(p, \mu^{t_i}(p)\right), x, q\right)$$

ii. If $\tilde{F}^*$ is commutative, it satisfies $\forall p, q \in Q, x, y \in \sum$ and let $i \geq 1$

$$\tilde{\delta}^*\left(\left(q, \mu^{t_{i-1}}(q)\right), xy, p\right) = \tilde{\delta}^*\left(\left(q, \mu^{t_{i-1}}(q)\right), yx, p\right)$$

If $\tilde{F}^*$ satisfies both switching and commutative, thus $\tilde{F}^*$ is called as **General Fuzzy Switchboard Automata (GFSA)**.

5.2.1 Illustrative example of General Fuzzy Switchboard Automata

Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a max-min general fuzzy automaton, where $Q = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{x, y\}, Q_{act}(t_0) = \{q_0, \mu^{t_0}(q_0)\} = \{(q_0, 1)\}, F_1(\mu, \delta) = \min(\mu, \delta)$, $Z = \phi$,ω and F_2 are not applicable. If the input string x_k is chosen, then

$$Q_{act}(t_0) = \{(q_0, \mu^{t_0}(q_0))\} = \{(q_0, 1)\}, v_i = \tilde{\delta}\left((q_i, \mu^{t_i}(q_i)), x_k, q_j\right)$$
$$= F_1\left(\mu^{t_i}(q_i), \delta(q_i, x_k, q_j)\right),$$

$$F_1(\mu, \delta) = \delta, F_2{}_{i=0}^{n-1}[v_i] = \mu^{t+1}(q_j) = \bigwedge_{i=0}^{n-1} (F_1\left(\mu^{t_i}(q_i), \delta(q_i, x_k, q_j)\right))$$

Where $n-1$ is the number of simultaneous transitions to the active state q_j at time $t+1$.

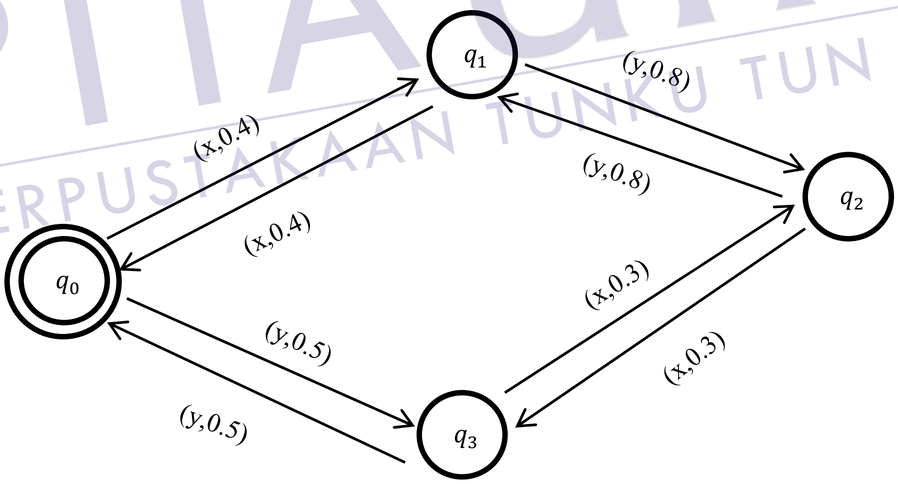

Figure 5.1. The example of GFSA

When string $a = xyxy$ then

$$\mu^{t_0}(q_0) = 1, \mu^{t_1}(q_1) = \tilde{\delta}\left((q_0, \mu^{t_0}(q_0)), x, q_1\right) = F_1(\mu^{t_0}(q_0), \delta(q_0, x, q_1))$$
$$= F_1(1{,}0.4) = 0.4,$$

$$\mu^{t_2}(q_2) = \tilde{\delta}\left((q_1, \mu^{t_1}(q_1)), y, q_2\right) = F_1(\mu^{t_1}(q_1), \delta(q_1, y, q_2)) = F_1(0.4{,}0.8) = 0.4,$$

$$\mu^{t_3}(q_3) = \tilde{\delta}\left((q_2, \mu^{t_2}(q_2)), x, q_3\right) = F_1\big(\mu^{t_2}(q_2), \delta(q_2, x, q_3)\big) = F_1(0.4,0.3) = 0.3,$$

$$\mu^{t_4}(q_0) = \tilde{\delta}\left((q_3, \mu^{t_3}(q_3)), y, q_0\right) = F_1\big(\mu^{t_3}(q_3), \delta(q_3, y, q_0)\big) = F_1(0.3,0.5) = 0.3.$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), x, q_1\right) = 0.4,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), xy, q_2\right) = 0.4 \wedge 0.4 = 0.4,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), xyx, q_4\right) = 0.4 \wedge 0.4 \wedge 0.3 = 0.3,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), xyxy, q_0\right) = 0.4 \wedge 0.4 \wedge 0.3 \wedge 0.3 = 0.3.$$

If the input string $a = yxyx$, then

$$\mu^{t_0}(q_0) = 1, \mu^{t_1}(q_3) = \tilde{\delta}\left((q_0, \mu^{t_0}(q_0)), y, q_3\right) = F_1\big(\mu^{t_0}(q_0), \delta(q_0, y, q_3)\big)$$
$$= F_1(1,0.5) = 0.5,$$

$$\mu^{t_2}(q_2) = \tilde{\delta}\left((q_3, \mu^{t_1}(q_3)), x, q_3\right) = F_1\big(\mu^{t_1}(q_3), \delta(q_3, x, q_2)\big) = F_1(0.5,0.3) = 0.3,$$

$$\mu^{t_3}(q_1) = \tilde{\delta}\left((q_2, \mu^{t_2}(q_2)), y, q_1\right) = F_1\big(\mu^{t_2}(q_2), \delta(q_2, y, q_1)\big) = F_1(0.3,0.8) = 0.3,$$

$$\mu^{t_4}(q_0) = \tilde{\delta}\left((q_1, \mu^{t_3}(q_0)), x, q_0\right) = F_1\big(\mu^{t_3}(q_1), \delta(q_1, x, q_0)\big) = F_1(0.3,0.4) = 0.3.$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), y, q_3\right) = 0.5,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), yx, q_2\right) = 0.5 \wedge 0.3 = 0.3,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), yxy, q_1\right) = 0.5 \wedge 0.3 \wedge 0.3 = 0.3,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), yxyx, q_0\right) = 0.5 \wedge 0.3 \wedge 0.3 \wedge 0.3 = 0.3.$$

Table 5.1 and Table 5.2 show the operation of the fuzzy automaton upon input string $xyxy$ and $yxyx$ for membership assignment functions and multi-membership resolution strategies.

Table 5.1: Active states and their membership values of $xyxy$

Time	Input	$Q_{act}(t_i)$	Membership value
t_0	Λ	q_0	1.0
t_1	x	q_1	0.4
t_2	y	q_2	0.4
t_3	x	q_3	0.3
t_4	y	q_4	0.3

Table 5.2: Active states and their membership values of $yxyx$

Time	Input	$Q_{act}(t_i)$	Membership value
t_0	Λ	q_0	1.0
t_1	y	q_3	0.5
t_2	x	q_2	0.3
t_3	y	q_1	0.3
t_4	x	q_0	0.3

Since the General Fuzzy Switchboard Automata (GFSA) is applied, the calculation for Figure 5.1 shows the membership values of active states are considered for each transition. Based on Figure 5.1, it can be seen directly that each state follows the switching properties. For instance, $(q_0, x, q_1) = (q_1, x, q_0)$. Calculating the membership value of the string $xyxy$ and $yxyx$ has tested the commutative property. Since $\tilde{\delta}^*\left(\left(q_0, \mu^{t_0}(q_0)\right), xyxy, q_0\right) = \tilde{\delta}^*\left(\left(q_0, \mu^{t_0}(q_0)\right), yxyx, q_0\right)$, Figure 5.1 is commutative. Thus, the example for Figure 5.1 is GFSA.

5.5.2 Examples of General Fuzzy Automata

The examples of General Fuzzy Automata are provided and some calculations are stated below. Throughout this section, some properties regarding GFSA are introduced.

Example 5.5.2.1:

Let $\tilde{F}^* = (Q, \Sigma, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a max-min general fuzzy automaton, where $Q = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{x, y\}$, $Q_{act}(t_0) = \{q_0, \mu^{t_0}(q_0)\} = \{(q_0, 1)\}$, $F_1(\mu, \delta) = \min(\mu, \delta)$, $Z = \phi$,ω and F_2 are not applicable. If the input string x_k is chosen, then

$$Q_{act}(t_0) = \left\{\left(q_0, \mu^{t_0}(q_0)\right)\right\} = \{(q_0, 1)\}, v_i = \tilde{\delta}\left(\left(q_i, \mu^{t_i}(q_i)\right), x_k, q_j\right)$$

$$= F_1\left(\mu^{t_i}(q_i), \delta\left(q_i, x_k, q_j\right)\right),$$

$$F_1(\mu, \delta) = \delta, F_2{}_{i=0}^{n-1}[v_i] = \mu^{t+1}\left(q_j\right) = \bigwedge_{i=0}^{n-1}\left(F_1\left(\mu^{t_i}(q_i), \delta\left(q_i, x_k, q_j\right)\right)\right)$$

Where $n - 1$ is the number of simultaneous transitions to the active state q_j at time $t + 1$.

If the input string $a = xyxy$ is chosen, then

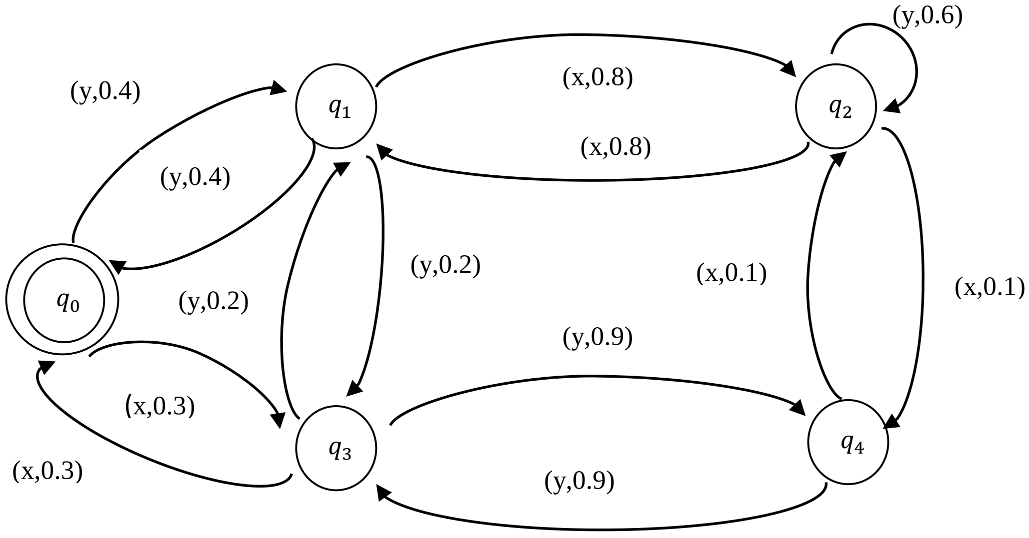

Figure 5.2: Switching state machine in GFA

$$\mu^{t_0}(q_0) = 1, \mu^{t_1}(q_3) = \tilde{\delta}\left((q_0, \mu^{t_0}(q_0)), x, q_3\right) = F_1\big(\mu^{t_0}(q_0), \delta(q_0, x, q_3)\big)$$

$$= F_1(1,0.3) = 0.3,$$

$$\mu^{t_2}(q_1) = \tilde{\delta}\left((q_3, \mu^{t_1}(q_3)), y, q_1\right) = F_1\big(\mu^{t_1}(q_3), \delta(q_3, y, q_1)\big) = F_1(0.3,0.2) = 0.2,$$

$$\mu^{t_2}(q_4) = \tilde{\delta}\left((q_3, \mu^{t_1}(q_0)), y, q_4\right) = F_1\big(\mu^{t_1}(q_3), \delta(q_3, y, q_4)\big) = F_1(0.3,0.9) = 0.3,$$

$$\mu^{t_3}(q_2) = \tilde{\delta}\left((q_1, \mu^{t_2}(q_1)), x, q_2\right) \wedge \tilde{\delta}\left((q_4, \mu^{t_2}(q_4)), x, q_2\right)$$

$$= F_1\big(\mu^{t_2}(q_1), \delta(q_1, x, q_2)\big) \wedge F_1\big(\mu^{t_2}(q_4), \delta(q_4, x, q_2)\big)$$

$$= F_1(0.2,0.8) \wedge F(0.3,0.1) = 0.2 \wedge 0.1 = 0.1.$$

$$\mu^{t_4}(q_2) = \tilde{\delta}\left((q_2, \mu^{t_3}(q_2)), y, q_2\right) = F_1\big(\mu^{t_3}(q_2), \delta(q_2, y, q_2)\big) = F_1(0.1,0.6) = 0.1,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), x, q_3\right) = 0.3,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), xy, q_1\right) = 0.3 \wedge 0.2 = 0.2,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), xyx, q_2\right) = 0.3 \wedge 0.2 \wedge 0.1 = 0.1,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), xyxy, q_2\right) = 0.3 \wedge 0.2 \wedge 0.1 \wedge 0.1 = 0.1.$$

If the input string $a = yxyx$ is chosen, then

$$\mu^{t_0}(q_0) = 1, \mu^{t_1}(q_1) = \tilde{\delta}\left((q_0, \mu^{t_0}(q_0)), y, q_1\right) = F_1(\mu^{t_0}(q_0), \delta(q_0, y, q_1))$$
$$= F_1(1{,}0.4) = 0.4,$$

$$\mu^{t_2}(q_2) = \tilde{\delta}\left((q_1, \mu^{t_1}(q_1)), x, q_2\right) = F_1(\mu^{t_1}(q_1), \delta(q_1, x, q_2)) = F_1(0.4{,}0.8) = 0.4,$$

$$\mu^{t_3}(q_2) = \tilde{\delta}\left((q_2, \mu^{t_1}(q_2)), y, q_2\right) = F_1(\mu^{t_2}(q_2), \delta(q_2, y, q_2)) = F_1(0.4{,}0.6) = 0.4,$$

$$\mu^{t_4}(q_4) = \tilde{\delta}\left((q_2, \mu^{t_3}(q_2)), x, q_4\right) = F_1(\mu^{t_3}(q_2), \delta(q_2, x, q_4)) = F_1(0.4{,}0.1) = 0.1,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), y, q_1\right) = 0.4,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), yxy, q_2\right) = 0.4 \wedge 0.4 \wedge 0.4 = 0.4,$$

$$\tilde{\delta}^*\left((q_0, \mu^{t_0}(q_0)), yxyx, q_4\right) = 0.4 \wedge 0.4 \wedge 0.4 \wedge 0.1 = 0.1.$$

Table 5.3 and Table 5.4 show the operation of the fuzzy automaton upon input string $xyxy$ and $yxyx$ for membership assignment functions and multi-membership resolution strategies.

Table 5.3: Active states and their membership values of $xyxy$ for Figure 5.3

Time	t_0	t_1	t_2		t_3	t_4
Input	Λ	x	y		x	y
$Q_{act}(t_i)$	q_0	q_3	q_1	q_4	q_2	q_2
Membership value	1.0	0.3	0.2	0.3	0.1	0.1

Table 5.4: Active states and their membership values of $yxyx$ for Figure 5.3

Time	Input	$Q_{act}(t_i)$	Membership value
t_0	Λ	q_0	1.0
t_1	y	q_1	0.4
t_2	x	q_2	0.4
t_3	y	q_2	0.4
t_4	x	q_4	0.1

Table 5.3 and Table 5.4, show that $\tilde{F}^*$ is switching but not commutative since $\delta^*((q, \mu^{t_{i-1}}(q))xy, p) \neq \delta^*((q, \mu^{t_{i-1}}(q))yx, p)$. Thus, this system is not switchboard.

Example 5.5.2.2:

Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a max-min general fuzzy automaton, where $Q = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{x, y\}$, $F_1(\mu, \delta) = \min(\mu, \delta)$, $Z = \phi$,ω and F_2 are not applicable. If the input string x_k is chosen, then

$$Q_{act}(t_0) = \{(q_0, \mu^{t_0}(q_0))\} = \{(q_0, 1)\}, v_i = \tilde{\delta}\left((q_i, \mu^{t_i}(q_i)), x_k, q_j\right)$$
$$= F_1\left(\mu^{t_i}(q_i), \delta(q_i, x_k, q_j)\right),$$

$$F_1(\mu, \delta) = \delta, F_2{}_{i=0}^{n-1}[v_i] = \mu^{t+1}(q_j) = \bigwedge_{i=0}^{n-1}(F_1\left(\mu^{t_i}(q_i), \delta(q_i, x_k, q_j)\right))$$

Where $n - 1$ is the number of simultaneous transitions to the active state q_j at time $t + 1$.

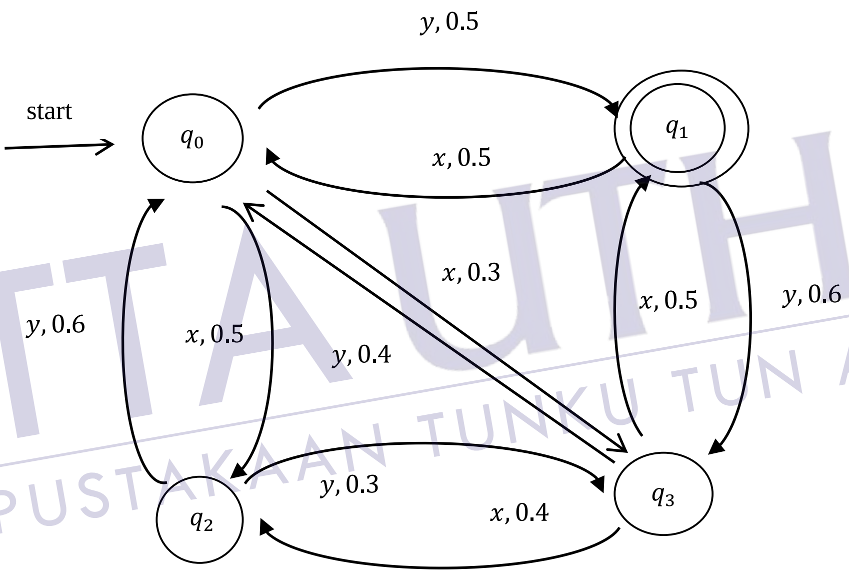

Figure 5.3. Non-switching state machine in GFA

$$\mu^{t_0}(q_0) = 1, \mu^{t_1}(q_1) = \tilde{\delta}\left((q_0, \mu^{t_0}(q_0)), y, q_1\right) = F_1(\mu^{t_0}(q_0), \delta(q_0, y, q_1))$$
$$= F_1(1{,}0.5) = 0.5,$$

$$\mu^{t_1}(q_2) = \tilde{\delta}\left((q_0, \mu^{t_0}(q_0)), x, q_2\right) = F_1(\mu^{t_0}(q_0), \delta(q_0, x, q_2)) = F_1(1{,}0.5) = 0.5,$$

$$\mu^{t_1}(q_3) = \tilde{\delta}\left((q_0, \mu^{t_0}(q_0)), x, q_3\right) = F_1(\mu^{t_0}(q_0), \delta(q_0, x, q_3)) = F_1(1{,}0.3) = 0.3,$$

$$\mu^{t_2}(q_3) = F_1(\mu^{t_1}(q_1), \delta(q_1, y, q_3)) \wedge F_1(\mu^{t_1}(q_2), \delta(q_2, y, q_3))$$
$$= F_1(0.5{,}0.6) \wedge F_1(0.5{,}0.3) = 0.5 \wedge 0.3 = 0.3,$$

$$\mu^{t_3}(q_1) = F_1(\mu^{t_2}(q_3), \delta(q_3, x, q_1)) = F_1(0.3{,}0.5) = 0.3,$$

Since $\delta^*\left(\left(q,\mu^{t_{i-1}}(q)\right)x,p\right) \neq \delta^*\left(\left(p,\mu^{t_{i-1}}(p)\right)x,q\right)$, then $\tilde{F}^*$ is not switching. Thus, this system is not switchboard.

Proposition 5.0:

Let $\breve{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton, if $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ is a commutative GFSA, then for every $i \geq 1$, $\tilde{\delta}^*((q,\mu^{t_{i-1}}(q), xa, p) = \tilde{\delta}^*((q,\mu^{t_{i-1}}(q), ax, p)$ for all $q \in Q_{act}(t_{i-1}), p \in S_c(q), a \in \sum$ and $x \in \sum^*$.

Proof:

Since $p \in S_c(q)$ then $q \in Q_{act}(t_{i-1})$ and $|x| = n$. If $n = 0$, then $x = \Lambda$. Thus

$$\tilde{\delta}^*((q,\mu^{t_{i-1}}(q), xa, p) = \tilde{\delta}^*((q,\mu^{t_{i-1}}(q), \Lambda a, p)$$

$$= \tilde{\delta}^*((q,\mu^{t_{i-1}}(q), a, p)$$

$$= \tilde{\delta}^*((q,\mu^{t_{i-1}}(q), a\Lambda, p)$$

$$= \tilde{\delta}^*((q,\mu^{t_{i-1}}(q), ax, p)$$

Suppose the result is true for all $u \in \sum^*$ with $|u| = n - 1$, where $n > 0$. Let $x = ub$, where $b \in \sum$. Then

$$\tilde{\delta}^*((q,\mu^{t_{i-1}}(q), xa, p) = \tilde{\delta}^*((q,\mu^{t_{i-1}}(q), uba, p)$$

$$= \bigvee_{r \in Q_{act}(t_i)} (\tilde{\delta}(\left(q,\mu^{t_{i-1}}(q)\right), u, r) \wedge \tilde{\delta}(\left(r,\mu^{t_i}(r)\right), ba, p)$$

$$= \bigvee_{r \in Q_{act}(t_i)} (\tilde{\delta}(\left(q,\mu^{t_{i-1}}(q)\right), u, r) \wedge \tilde{\delta}(\left(r,\mu^{t_i}(r)\right), ab, p)$$

$$= \tilde{\delta}^*((q,\mu^{t_{i-1}}(q), uab, p)$$

$$\bigvee_{r \in Q_{act}(t_i)} (\tilde{\delta}(\left(q,\mu^{t_{i-1}}(q)\right), au, r) \wedge \tilde{\delta}(\left(r,\mu^{t_i}(r)\right), b, p)$$

$$= \tilde{\delta}^*((q,\mu^{t_{i-1}}(q), aub, p)$$

$$= \tilde{\delta}^*((q,\mu^{t_{i-1}}(q), ax, p)$$

This proposition 5.0 completes the proof.

Proposition 5.1:

Let $\breve{F} = (Q, \Sigma, \widetilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton, if $\tilde{F}^* = (Q, \Sigma, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ is a switching GFSA, then for every $i \geq 0$, $\tilde{\delta}^*((q, \mu^{t_{i-1}}(q), x, p) = \tilde{\delta}^*((p, \mu^{t_{i-1}}(p), x, q)$ for all $p, q \in Q_{act}(t_i)$ and $x \in \sum^*$.

Proof:

Since $p, q \in Q_{act}(t_i)$ and $x \in \sum^*$, prove the result by induction on $|x| = n$. First, assume that $x = \Lambda$, whenever $n = 0$. Then $\tilde{\delta}^*((q, \mu^{t_{i-1}}(q), x, p) = \tilde{\delta}^*((q, \mu^{t_{i-1}}(q), \Lambda, p) = \tilde{\delta}^*((p, \mu^{t_{i-1}}(p), \Lambda, q) = \tilde{\delta}^*((p, \mu^{t_{i-1}}(p), x, q)$. Thus, the theorem holds for $x = \Lambda$. Now, assume the result holds for all $u \in \sum^*$ such that $|u| = n - 1$ and $n > 0$. Let $a \in \sum$ and $x \in \sum^*$ be such that $x = ua$. Then

$$\tilde{\delta}^*((q, \mu^{t_i}(q), x, p) = \tilde{\delta}^*((q, \mu^{t_i}(q), ua, p)$$

$$= \bigvee_{r \in Q_{act}(t_{i+1})} (\tilde{\delta}((q, \mu^{t_i}(q)), u, r) \wedge \tilde{\delta}((r, \mu^{t_{i+1}}(r)), a, p)$$

$$= \bigvee_{r \in Q_{act}(t_i)} (\tilde{\delta}((r, \mu^{t_i}(r)), u, q) \wedge \tilde{\delta}((p, \mu^{t_{i+1}}(p)), a, r)$$

$$= \bigvee_{r \in Q_{act}(t_{i+1})} (\tilde{\delta}((p, \mu^{t_i}(p)), a, r) \wedge \tilde{\delta}((r, \mu^{t_{i+1}}(r)), u, q)$$

$$= \tilde{\delta}^*((p, \mu^{t_i}(p), au, q)$$

$$= \tilde{\delta}^*((p, \mu^{t_i}(p), ua, q)$$

$$= \tilde{\delta}^*((p, \mu^{t_i}(p), x, q)$$

Hence, the result is true for on $|u| = n$. This completes the proof for proposition 5.1.

Proposition 5.2:

Let $\breve{F} = (Q, \Sigma, \widetilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton, if $\tilde{F}^* = (Q, \Sigma, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ is a GFSA, then for every $i \geq 1$, $\tilde{\delta}^*((q, \mu^{t_{i-1}}(q), xy, p) = \tilde{\delta}^*((p, \mu^{t_{i-1}}(p), yx, q)$ for all $p, q \in Q$ and $x, y \in \sum^*$.

Proof:

Since $p, q \in Q$ and $x, y \in \sum^*$, prove the result by induction on $|x| = n$. Firstly, assume that $n = 0$, the $y = \Lambda$. Thus,

$$\tilde{\delta}^*((q,\mu^{t_{i-1}}(q),xy,p) = \tilde{\delta}^*((q,\mu^{t_{i-1}}(q),x\Lambda,p)$$

$$= \tilde{\delta}^*((q,\mu^{t_{i-1}}(q),\Lambda\mathrm{x},p)$$

$$= \tilde{\delta}^*((q,\mu^{t_{i-1}}(q),yx,p)$$

Suppose that $\tilde{\delta}^*((q,\mu^{t_{i-1}}(q),xu,p) = \tilde{\delta}^*((p,\mu^{t_{i-1}}(p),ux,q)$ for every $u \in \sum^*$.

Now, continue the result is true for all $u \in \sum^*$ with $|u| = n-1$, where $n > 0$. Let $y = ua$, where $a \in \sum$ and $u \in \sum^*$. Then

$$\tilde{\delta}^*((q,\mu^{t_{i-1}}(q),xy,p) = \tilde{\delta}^*((q,\mu^{t_{i-1}}(q),xua,p)$$

$$= \bigvee_{r\in Q_{act}(t_i)} (\tilde{\delta}((q,\mu^{t_{i-1}}(q)),xu,r) \wedge \tilde{\delta}((r,\mu^{t_i}(r)),a,p)$$

$$= \bigvee_{r\in Q_{act}(t_i)} (\tilde{\delta}((q,\mu^{t_{i-1}}(q)),ux,r) \wedge \tilde{\delta}((r,\mu^{t_i}(r)),a,p)$$

$$= \bigvee_{r\in Q_{act}(t_{i-1})} (\tilde{\delta}((r,\mu^{t_{i-1}}(r)),ux,q) \wedge \tilde{\delta}((p,\mu^{t_i}(p)),a,r)$$

$$= \bigvee_{r\in Q_{act}(t_i)} (\tilde{\delta}((p,\mu^{t_{i-1}}(p)),a,r) \wedge \tilde{\delta}((r,\mu^{t_i}(r)),ux,q)$$

$$= \tilde{\delta}^*((p,\mu^{t_{i-1}}(p),aux,q)$$

$$= \bigvee_{r\in Q_{act}(t_i)} (\tilde{\delta}((p,\mu^{t_{i-1}}(p)),au,r) \wedge \tilde{\delta}((r,\mu^{t_i}(r)),x,q)$$

$$= \bigvee_{r\in Q_{act}(t_i)} (\tilde{\delta}((p,\mu^{t_{i-1}}(p)),ua,r) \wedge \tilde{\delta}((r,\mu^{t_i}(r)),x,q)$$

$$= \tilde{\delta}^*((p,\mu^{t_{i-1}}(p),uax,q)$$

$$= \tilde{\delta}^*((q,\mu^{t_i}(q),uax,p)$$

$$= \tilde{\delta}^*((q,\mu^{t_i}(q),yx,p)$$

The proposition 5.2 is proven.

Theorem 5.0:

Let $\breve{F} = (Q,\Sigma,\widetilde{R},Z,\tilde{\delta},\omega,F_1,F_2)$ be a general fuzzy automaton, if $\tilde{F}^* = (Q,\Sigma,\widetilde{R},Z,\tilde{\delta}^*,\omega,F_1,F_2)$ is a max-min fuzzy automaton. Define a relation $\equiv$ on X by $x \equiv y\ \forall x,y \in \sum^*$ if and only if $\tilde{\delta}^*((q,\mu^{t_i}(q),x,p) = \tilde{\delta}^*((q,\mu^{t_i}(q),y,p)\ \forall p,q \in Q$. Then the relation $\equiv$ is congruence relation on $\sum^*$.

Proof:

Obviously $\equiv$ is an equivalence relation on $\sum^*$. Let $x, y, z \in \sum^*$ be such that $x \equiv y$ and let $p, q \in Q$. Then

$$\tilde{\delta}^*((q, \mu^{t_i}(q), xz, p) = \bigvee_{r \in Q} (\tilde{\delta}((q, \mu^{t_i}(q)), x, r) \wedge \tilde{\delta}((r, \mu^{t_{i+1}}(r)), z, p)$$

$$= \bigvee_{r \in Q} (\tilde{\delta}((q, \mu^{t_i}(q)), y, r) \wedge \tilde{\delta}((r, \mu^{t_{i+1}}(r)), z, p)$$

$$= \tilde{\delta}^*((q, \mu^{t_i}(q), yz, p)$$

Hence $xz \equiv yz$. Similarly $zx \equiv zy$. Therefore $\equiv$ is a congruence relation on $\sum^*$.

Now let $x \in \sum^*$ and $x = x_1 x_2 \cdots c_n$ where $x_1, x_2, \cdots, x_n \in X$. For every $p, q \in Q$, $\tilde{\delta}^*\left(\left(q, \mu^{t_i}(q)\right), x, p\right) =$

$\vee_{r_1, r_2, \cdots, r_{n-1} \in Q}((\tilde{\delta}\left(\left(q, \mu^{t_i}(q)\right), x_1, r_1\right) \wedge \tilde{\delta}\left(\left(r_1, \mu^{t_{i+1}}(r_1)\right), x_2, r_2\right) \wedge \cdots \wedge$

$\tilde{\delta}(\left(r_{n-1}, \mu^{i+n}(r_{n-1})\right), x_n, p))$ is obtained.

Since the image $\tilde{\delta}$ is finite, the image of $\tilde{\delta}^*$ is also a finite. Hence the following theorem is introduced.

Theorem 5.1:

Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ is a max-min fuzzy automaton. Define a binary operation $\odot$ on $S(\tilde{F}^*)$ by $[x] \odot [y] = [xy]$ for all $x, y \in S(\tilde{F}^*)$. Then $(S(\tilde{F}^*), \odot)$ is a finite semigroup with identity.

Proof:

It is clear that the binary operation $\odot$ is associative. Now, show that λ is the identity of $(S(\tilde{F}^*), \odot)$. Then, $[x] \odot [\lambda] = [x\lambda] = [x] = [\lambda x] = [\lambda] \odot [x]$. Hence $(S(\tilde{F}^*), \odot)$ is a finite semigroup with identity.

Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ is a max-min fuzzy automaton and let $\sim$ be an equivalence relation on Q. For $q \in Q$, let $[q]$ denote the equivalence class of q. Let $Q_\sim = Q/\sim = \{[q] | q \in Q\}$. Define the fuzzy subset $\delta_\sim^*$ of $(Q_\sim \times [0,1]) \times \sum \times Q_\sim$ by

$$\tilde{\delta}_\sim^*([q], x, [p]) = \vee \left\{\tilde{\delta}^*\left(\left(a, \mu^{t_i}(a)\right), x, b\right) | a \in [q], b \in [p]\right\}$$

For all $p, q \in Q, x \in \sum. \tilde{F}^*{}_{\sim} = (Q_{\sim}, \Sigma, \tilde{R}_{\sim}, Z, \tilde{\delta}^*_{\sim}, \omega, F_1, F_2)$ is max-min fuzzy automaton when $\tilde{\delta}^*_{\sim}$ is single-valued.

Theorem 5.2:

Let $\tilde{F}^* = (Q, \sum, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ is a GFSA and let $\sim$ be an equivalence relation on Q then $\tilde{F}^*{}_{\sim} = (Q_{\sim}, \Sigma, \tilde{R}_{\sim} Z, \tilde{\delta}^*{}_{\sim}, \omega, F_1, F_2)$ is a GFSA.

Proof:

Since $\tilde{F}^*$ is switching, then $\forall [q], [p] \in Q_{\sim}$ and $u \in \Sigma$,

$$\tilde{\delta}^*{}_{\sim}\left(\left([q], \mu^{t_i}([q])\right), u, [p]\right) = \vee\left\{\tilde{\delta}^*\left(\left(a, \mu^{t_i}(a)\right), u, b\right) \middle| a \in [q], b \in [p]\right\}$$

$$= \vee\left\{\tilde{\delta}^*\left(\left(b, \mu^{t_i}(b)\right), u, a\right) \middle| a \in [q], b \in [p]\right\}$$

$$= \tilde{\delta}^*{}_{\sim}\left(\left([p], \mu^{t_i}([p])\right), u, [q]\right)$$

Thus $\tilde{F}^*{}_{\sim}$ is switching.

Since $\tilde{F}^*$ is commutative, then $\forall u, v \in \sum$

$$\tilde{\delta}^*{}_{\sim}\left(\left([q], \mu^{t_{i+j}}([q])\right), uv, [p]\right)$$

$$= \vee\{\tilde{\delta}^*\left(\left([q], \mu^{t_i}([q])\right), u, [r]\right) \wedge \tilde{\delta}^*\left(\left([r], \mu^{t_j}([r])\right), v, [p]\right) | [r] \in Q_{\sim}\}$$

$$= \bigvee\{\vee\{\tilde{\delta}^*\left(\left(a, \mu^{t_i}(a)\right), u, c\right) | a \in [q], c \in [r]\}) \wedge (\vee\{\tilde{\delta}^*\left(\left(c, \mu^{t_j}(c)\right), v, b\right) | c \in [r], b \in [p]\}) | [r] \in Q_{\sim}\}$$

$$= \bigvee\{\vee\left\{\tilde{\delta}^*\left(\left(a, \mu^{t_i}(a)\right), u, c\right) \wedge \tilde{\delta}^*\left(\left(c, \mu^{t_j}(c)\right), v, b\right) \middle| c \in [r], [r] \in Q_{\sim}\right\} | a \in [q], b \in [p]\}$$

$$= \vee\left\{\tilde{\delta}^*\left(\left(a, \mu^{t_{i+j}}(a)\right), uv, b\right) \middle| a \in [q], b \in [p]\right\}$$

$$= \vee\left\{\tilde{\delta}^*\left(\left(a, \mu^{t_{i+j}}(a)\right), vu, b\right) \middle| a \in [q], b \in [p]\right\}$$

$$= \bigvee\{\vee\left\{\tilde{\delta}^*\left(\left(a, \mu^{t_i}(a)\right), v, c\right) \wedge \tilde{\delta}^*\left(\left(c, \mu^{t_j}(c)\right), u, b\right) \middle| c \in [r], [r] \in Q_{\sim}\right\} | a \in [q], b \in [p]\}$$

$$= \bigvee\{\vee\{\tilde{\delta}^*\left((a,\mu^{t_i}(a)),v,c\right)|a \in [q], c \in [r]\})\wedge(\vee\{\tilde{\delta}^*\left((c,\mu^{t_j}(c)),u,b\right)|c \in [r], b \in [p]\})|[r] \in Q_{\sim}\}$$

$$= \vee\{\tilde{\delta}^*\left(([q],\mu^{t_i}([q])),v,[r]\right) \wedge \tilde{\delta}^*\left(([r],\mu^{t_j}([r])),u,[p]\right)|[r] \in Q_{\sim}\}$$

$$= \tilde{\delta}^*{}_{\sim}\left(([q],\mu^{t_{i+j}}([q])),vu,[p]\right)$$

Thus $\tilde{F}^*{}_{\sim}$ is commutative. Therefore, $\tilde{F}^*{}_{\sim} = (Q_{\sim},\Sigma,\tilde{R}_{\sim}Z,\tilde{\delta}^*{}_{\sim},\omega,F_1,F_2)$ is a GFSA.

Definition 5.1:

Let $\tilde{F}^* = (Q,\sum,\tilde{R},Z,\tilde{\delta}^*,\omega,F_1,F_2)$ is a GFSA. If $\sigma \in \sum$ exists such that $\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\sigma,p\right) = \tilde{\delta}^*\left((p,\mu^{t_i}(p)),\sigma,q\right) \forall p,q \in Q_{act}(t_i)$, q and p are in a switching relation, then denote that a switching class such as $[q]_\sigma = \left\{p\middle|\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\sigma,p\right) = \tilde{\delta}^*\left((p,\mu^{t_i}(p)),\sigma,q\right), \sigma \in \Sigma\right\}$.

Proposition 5.3:

Let $\tilde{F}^* = (Q,\sum,\tilde{R},Z,\tilde{\delta}^*,\omega,F_1,F_2)$ be a GFSA. Define $Q_\sigma = \{[q_\sigma]|q \in Q\}$ and $\sum_\sigma = X\{\sigma\}$ for $\sigma \in \sum$. Take $\alpha = [q]_\sigma, q \in Q_{act}(t_i)$. For all $\tau \in \sum_\sigma$, define the fuzzy subset $\tilde{\delta}^*(\alpha,\tau,\beta) = \tilde{\delta}^*\left(([q]_\sigma,\mu^{t_i}([q]_\sigma)),\tau,[p]_\sigma\right) = \vee\left\{\tilde{\delta}^*\left((r,\mu^{t_i}(r)),\tau,s\right): r \in [q]_\sigma, s \in [p]_\sigma\right\} \forall \alpha,\beta \in Q_\sigma$, then $\tilde{F}^*{}_\sigma = (Q_\sigma,\sum_\sigma,\tilde{R}_\sigma,Z_\sigma,\tilde{\delta}^*{}_\sigma,\omega_\sigma,F_1,F_2)$ is a GFSA.

Proof:

Let $\alpha,\beta \in Q_\sigma$ and take $\alpha = [q]_\sigma, \beta = [p]_\sigma, \forall p,q \in Q_{act}(t_i)$. Suppose that $\alpha = \beta$. Then $\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\sigma,p\right) = \tilde{\delta}^*\left((p,\mu^{t_i}(p)),\sigma,q\right) \Rightarrow p = q$. Now, assume that $\tau = \rho, \forall \tau,\rho \in \sum_\sigma$.

Case 1(a): if $q = p$, then

$$\tilde{\delta}^*{}_\sigma(\alpha,\tau,\beta) = \tilde{\delta}^*\left(([q]_\sigma,\mu^{t_i}([q]_\sigma)),\tau,[p]_\sigma\right)$$

$$= \tilde{\delta}^*\left(([q]_\sigma,\mu^{t_i}([q]_\sigma)),\rho,[p]_\sigma\right)$$

$$= \tilde{\delta}^*\left(([p]_\sigma,\mu^{t_i}([p]_\sigma)),\rho,[q]_\sigma\right)$$

$$= \tilde{\delta}^*{}_\sigma(\beta,\rho,\alpha).$$

Case 1(b): if $\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\sigma,p\right)=\tilde{\delta}^*\left((p,\mu^{t_i}(p)),\sigma,q\right)$, then

$$\tilde{\delta}^*{}_{\sigma}(\alpha,\tau,\beta)=\tilde{\delta}^*\left(([q]_\sigma,\mu^{t_i}([q]_\sigma)),\tau,[p]_\sigma\right)$$

$$=\tilde{\delta}^*\left(([q]_\sigma,\mu^{t_i}([q]_\sigma)),\rho,[p]_\sigma\right)$$

$$=\bigwedge\{\tilde{\delta}^*\left((r,\mu^{t_i}(r)),\rho,s\right)$$

$$=\bigwedge\{\tilde{\delta}^*\left((s,\mu^{t_i}(s)),\rho,r\right),$$

$$=\tilde{\delta}^*\left(([p]_\sigma,\mu^{t_i}([p]_\sigma)),\rho,[q]_\sigma\right)$$

$$=\tilde{\delta}^*_{\sigma}(\beta,\rho,\alpha).$$

Thus, $\tilde{\delta}^*_{\sigma}$ is well-defined.

Let $\alpha,\beta\in Q_\sigma, r\in X_\sigma$ and take $\alpha=[q]_\sigma,\beta=[p]_\sigma$. Suppose that $\alpha=\beta$, then $\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\tau,p\right)=\tilde{\delta}^*\left((p,\mu^{t_i}(p)),\tau,q\right)$ or that $\alpha=\beta$. Then $\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\tau\sigma,p\right)=\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\sigma\tau,p\right)$.

Case 2(a): if $\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\tau,p\right)=\tilde{\delta}^*\left((p,\mu^{t_i}(p)),\tau,q\right)$, then

$$\tilde{\delta}^*{}_{\sigma}(\alpha,\tau,\beta)=\tilde{\delta}^*\left(([q]_\sigma,\mu^{t_i}([q]_\sigma)),\tau,[p]_\sigma\right)$$

$$=\bigwedge\{\tilde{\delta}^*\left((r,\mu^{t_i}(r)),\tau,s\right)|r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$=\bigwedge\{\tilde{\delta}^*\left((s,\mu^{t_i}(s)),\tau,r\right)|r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$=\tilde{\delta}^*\left(([p]_\sigma,\mu^{t_i}([p]_\sigma)),\tau,[q]_\sigma\right)$$

$$=\tilde{\delta}^*{}_{\sigma}(\beta,\tau,\alpha).$$

Case 2(b): if $\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\tau\sigma,p\right)=\tilde{\delta}^*\left((q,\mu^{t_i}(q)),\sigma\tau,p\right)$, then

$$\tilde{\delta}^*{}_{\sigma}(\alpha,\tau\sigma,\beta)=\tilde{\delta}^*\left(([q]_\sigma,\mu^{t_i}([q]_\sigma)),\tau\sigma,[p]_\sigma\right)$$

$$=\bigvee\{\tilde{\delta}^*\left((r,\mu^{t_i}(r)),\tau\sigma,s\right)|r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$=\bigvee\{\tilde{\delta}^*\left((r,\mu^{t_i}(r)),\tau,k\right)\wedge\tilde{\delta}^*\left((k,\mu^{t_{i+1}}(k)),\sigma,s\right)|k\in[t]_\sigma,r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$= \bigvee\{\tilde{\delta}^*\left(\left(k,\mu^{t_i}(k)\right),\tau,r\right)\wedge\tilde{\delta}^*\left(\left(s,\mu^{t_{i+1}}(s)\right),\sigma,k\right)|k\in[t]_\sigma,r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$= \bigvee\{\tilde{\delta}^*\left(\left(s,\mu^{t_i}(s)\right),\sigma,k\right)\wedge\tilde{\delta}^*\left(\left(k,\mu^{t_{i+1}}(k)\right),\tau,r\right)|k\in[t]_\sigma,r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$= \bigvee\{\tilde{\delta}^*\left(\left(s,\mu^{t_i}(s)\right),\sigma\tau,r\right)|r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$= \bigvee\{\tilde{\delta}^*\left(\left(s,\mu^{t_i}(s)\right),\tau\sigma,r\right)|r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$= \tilde{\delta}^*\left(\left([p]_\sigma,\mu^{t_i}([p]_\sigma)\right),\tau\sigma,[q]_\sigma\right)$$

$$= \tilde{\delta}^*_\sigma(\beta,\tau\sigma,\alpha).$$

Hence $\tilde{F}^*_\sigma$ is switching.

Let $\alpha,\beta\in Q_\sigma,\rho,\tau\in\sum_\sigma$ and take $\alpha=[q]_\sigma,\beta=[p]_\sigma,\forall p,q\in Q$. Since $\tilde{F}^*_\sigma$ is commutative, therefore

$$\tilde{\delta}^*{}_\sigma(\alpha,\tau\rho,\beta)=\tilde{\delta}^*\left(\left([q]_\sigma,\mu^{t_i}([q]_\sigma)\right),\tau\rho,[p]_\sigma\right)$$

$$= \bigvee\{\tilde{\delta}^*\left(\left([q]_\sigma,\mu^{t_i}([q]_\sigma)\right),\tau,[t]_\sigma\right)\wedge\tilde{\delta}^*\left(\left([t]_\sigma,\mu^{t_i}([t]_\sigma)\right),\rho,[p]_\sigma\right)|t\in Q\}$$

$$= \bigvee\{(\vee\{\tilde{\delta}^*\left(\left(r,\mu^{t_i}(r)\right),\tau,t\right)|t\in[t]_\sigma,r\in[q]_\sigma\}\wedge(\vee\{\tilde{\delta}^*\left(\left(t,\mu^{t_{i+1}}(t)\right),\rho,s\right)|t$$
$$\in[t]_\sigma,s\in[p]_\sigma\}|t\in Q\}$$

$$= \bigvee\{\tilde{\delta}^*\left(\left(r,\mu^{t_i}(r)\right),\tau,t\right)\wedge\tilde{\delta}^*\left(\left(t,\mu^{t_{i+1}}(t)\right),\rho,s\right)|t\in[t]_\sigma,t\in Q\}|r\in[q]_\sigma,s$$
$$\in[p]_\sigma\}$$

$$= \bigvee\{\tilde{\delta}^*\left(\left(r,\mu^{t_i}(r)\right),\tau\rho,s\right)|r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$= \bigvee\{\tilde{\delta}^*\left(\left(r,\mu^{t_i}(r)\right),\rho\tau,s\right)|r\in[q]_\sigma,s\in[p]_\sigma\}$$

$$= \bigvee\{\tilde{\delta}^*\left(\left(r,\mu^{t_i}(r)\right),\rho,t\right)\wedge\tilde{\delta}^*\left(\left(t,\mu^{t_{i+1}}(t)\right),\tau,s\right)|t\in[t]_\sigma,t\in Q\}|r\in[q]_\sigma,s$$
$$\in[p]_\sigma\}$$

$$= \bigvee\{(\vee\{\tilde{\delta}^*\left(\left(r,\mu^{t_i}(r)\right),\rho,t\right)|t\in[t]_\sigma,r\in[q]_\sigma\}\wedge(\vee\{\tilde{\delta}^*\left(\left(t,\mu^{t_{i+1}}(t)\right),\tau,s\right)|t$$
$$\in[t]_\sigma,s\in[p]_\sigma\}|t\in Q\}$$

$$= \bigvee\{\tilde{\delta}^*\left(\left([q]_\sigma,\mu^{t_i}([q]_\sigma)\right),\rho,[t]_\sigma\right)\vee\tilde{\delta}^*\left(\left([t]_\sigma,\mu^{t_i}([t]_\sigma)\right),\tau,[p]_\sigma\right)|t\in Q\}$$

$$= \tilde{\delta}^*\left(\left([q]_\sigma,\mu^{t_i}([q]_\sigma)\right),\rho\tau,[p]_\sigma\right)$$

$$= \tilde{\delta}^*_\sigma(\alpha, \rho\tau, \beta).$$

Hence $\tilde{F}^*_\sigma$ is commutative.

Note that, now $\tilde{F}^*_\sigma = (Q_\sigma, \sum_\sigma, \tilde{R}_\sigma, Z_\sigma, \tilde{\delta}^*{}_\sigma, \omega_\sigma, F_1, F_2)$ can be called s a quotient GFSA for $\sigma \in \sum$.

Definition 5.2:

Let $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ is a GFA. $\tilde{F}^*$ is said to be retrievable if for all $q \in Q, y \in \sum^*$ if $\exists t \in Q$ such that $\tilde{\delta}^*\left(\left(q, \mu^{t_j}(q)\right), y, t\right) > 0$, then $\exists x \in \sum^*$ such that $\tilde{\delta}^*\left(\left(t, \mu^{t_j}(t)\right), x, q\right) > 0$.

Theorem 5.3:

Let $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ is a strongly connected GFSA. If $\tilde{F}^*$ is equivalent for all $q \in Q$, then $\tilde{F}^*$ is retrievable GFA.

Proof:

Let for all $q, t \in Q$ and $y \in \sum^*$. Since $\tilde{F}^*$ is GFSA and strongly connected, then

$$\tilde{\delta}^*\left(\left(q, \mu^{t_j}(q)\right), y, t\right) > 0.$$

Since $\tilde{F}^*$ is equivalent, then x is equivalent to y for all $x, y \in \sum^*$, then

$$\tilde{\delta}^*\left(\left(q, \mu^{t_j}(q)\right), y, t\right) = \tilde{\delta}^*\left(\left(t, \mu^{t_j}(t)\right), x, q\right) > 0.$$

Since $\tilde{F}^*$ is switching,

$$\tilde{\delta}^*\left(\left(q, \mu^{t_j}(q)\right), x, t\right) = \tilde{\delta}^*\left(\left(t, \mu^{t_j}(t)\right), x, q\right) > 0.$$

Thus $\tilde{F}^*$ is retrievable on general fuzzy automata.

5.3 Switchboard subsystem and strong switchboard subsystem

A subsystem is designated together to perform a major part in the system. Each subsystem has its own specific function. As mentioned before in Chapter 1, one of the roles of the switchboard state machine is it can act as a controller to any two subsystems. Thus, this section investigated the switchboard subsystem in GFSA. The definition and properties of the switchboard subsystem and strong switchboard subsystem are provided.

Definition 5.3:

Let $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ is a GFSA. Let μ be a fuzzy subset of Q. Then μ is a switchboard subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every j, $1 \leq j \leq k$ such that $\mu^{t_j}(p) \geq \tilde{\delta}^*((q, \mu^{t_i}(q), x, p), \forall q, p \in Q$ and $x \in \sum$.

Theorem 5.4:

Let $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a GFSA and let μ be a fuzzy subset of Q. Then μ is a switchboard subsystem of $\tilde{F}^*$ if and only if $\mu^{t_j}(p) \geq \tilde{\delta}^*((q, \mu^{t_i}(q), x, p), \forall q \in Q_{(act)}(t_j), p \in Q$ and $x \in \sum^*$.

Proof:

Suppose that μ is a switchboard subsystem of $\tilde{F}^*$. Let $q \in Q_{(act)}(t_j), p \in Q$ and $x \in \sum^*$. The proof is by induction on $|x| = n$. If $n = 0$, then $x = \Lambda$. Now if $q = p$, then $\tilde{\delta}^*((p, \mu^{t_i}(p), \Lambda, p) = F_1\left(\mu^{t_i}(p), \tilde{\delta}(p, \Lambda, p)\right) = \mu^{t_i}(p)$. If $q \neq p$, then $\tilde{\delta}^*((q, \mu^{t_i}(q), \Lambda, p) = F_1\left(\mu^{t_i}(q), \tilde{\delta}(q, \Lambda, p)\right) = 0 < \mu^{t_j}(p)$.

Hence the result is true for $n = 0$. For now, assume that the result is valid for all $y \in \sum^*$ with $|y| = n - 1$, $n > 0$. For the y above, let $x = u_1 \cdots u_n$ where $u_i \in \Sigma, i = 1,2, \cdots, n$. Then

$$\tilde{\delta}^*((q, \mu^{t_i}(q), x, p) = \tilde{\delta}^*((q, \mu^{t_i}(q), u_1 \cdots u_n, p)$$

$$= \vee(\tilde{\delta}^*((q, \mu^{t_i}(q), u_1, r_1) \wedge \cdots \wedge \tilde{\delta}^*((r_{n-1}, \mu^{t_{i+n}}(r_{n-1})), u_n, p))$$

where $r_1 \in Q_{S(act)}(t_{i+1}) \cdots r_{n-1} \in Q_{S(act)}(t_{i+n})$

$$\leq \vee(\tilde{\delta}^*((r_{n-1}, \mu^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{1(act)}(t_{i+n}))$$

$$\leq \vee \mu^{t_j}(p)$$

$$= \mu^{t_j}(p)$$

Hence $\mu^{t_j}(p) \geq \tilde{\delta}^*((q, \mu^{t_i}(q), x, p)$.the converse is trivial. It is clear that μ satisfies switching and commutative, since $\tilde{F}^*$ is GFSA. This completes the proof.

Definition 5.4:

Let $\tilde{F}^* = (Q, \sum, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a GFSA. Let μ be a fuzzy subset of Q. Then μ is a strong switchboard subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every i, $1 \leq i \leq k$ such that $\tilde{\delta}^*((q, \mu^{t_i}(q), x, p) > 0$, then for $q, p \in Q$ and $x \in \sum, \mu^{t_j}(p) \geq \mu^{t_j}(q)$, for every $1 \leq j \leq k$.

Theorem 5.5:

Let $\tilde{F}^* = (Q, \sum, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a GFSA and let μ be a fuzzy subset of Q. Then μ is a strong switchboard subsystem of $\tilde{F}^*$ if and only if there exists $x \in \sum^*$such that $\tilde{\delta}^*((q, \mu^{t_i}(q), x, p) > 0$, then $\mu^{t_j}(p) \geq \mu^{t_j}(q)$, for all for $q \in Q_{(act)}(t_j), p \in Q$.

Proof:

Suppose that μ is a switchboard subsystem of $\tilde{F}^*$. Let $q \in Q_{(act)}(t_j), p \in Q$ and $x \in \sum^*$. The proof is by induction on $|x| = n$. If $n = 0$, then $x = \Lambda$. Now if $q = p$, then $\delta^*((p, \mu^{t_i}(p), \Lambda, p) = 1$ and $\mu^{t_j}(p) = \mu^{t_j}(q)$. If $q \neq p$, then $\tilde{\delta}^*((q, \mu^{t_i}(q), \Lambda, p) = F_1$ $\left(\mu^{t_i}(q), \tilde{\delta}(q, \Lambda, p)\right) = 0 < \mu^{t_j}(p)$. Hence the result is true for $n = 0$. For now, assume that the result is valid for all $u \in \sum^*$ with $|u| = n - 1$, $n > 0$. For the u above, let $x = u_1 \cdots u_n$ where $u_i \in \Sigma, i = 1,2, \cdots, n$. Suppose that $\tilde{\delta}^*((q, \mu^{t_i}(q), \mathrm{x}, p) > 0$. Then

$$\tilde{\delta}^*((q, \mu^{t_i}(q), u_1 \cdots u_n, p)$$
$$= \vee\{\tilde{\delta}^*((q, \mu^{t_i}(q), u_1, p_1) \wedge \cdots \wedge \tilde{\delta}^*\left((p_{n-1}, \mu^{t_{i+n}}(p_{n-1})), u_n, p)\right)\} > 0$$

where $p_1 \in Q_{(act)}(t_i) \cdots p_{n-1} \in Q_{(act)}(t_{i+n})$.

This implies that $\tilde{\delta}^*((q, \mu^{t_i}(q), u_1, p_1) > 0, \cdots, \tilde{\delta}^*((p_{n-1}, \mu^{t_{i+n}}(p_{n-1})), u_n, p) > 0$. Hence, $\mu^{t_j}(p) \geq \mu^{t_{i+n}}(p_{n-1}), \mu^{i+n}(p) \geq \mu^{t_{i+n-1}}(p_{n-2}), \cdots, \mu^{t_i}(p_1) \geq \mu^{t_j}(q)$. Thus, $\mu^{t_j}(p) \geq \mu^{t_j}(q)$. The converse is trivial. It is clear that μ satisfies switching and commutative, since $\tilde{F}^*$ is GFSA. This completes the proof.

Theorem 5.6:

Let $\tilde{F}^* = (Q, \sum, \widetilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a GFSA and let μ be a fuzzy subset of Q. If μ is a switchboard subsystem of $\tilde{F}^*$, then μ is a strong switchboard subsystem of $\tilde{F}^*$.

Proof:

Assume that $\tilde{\delta}^*((q, \mu^{t_i}(q), x, p) > 0 \ \forall x \in \sum$. Since μ is a switchboard subsystem of $\tilde{F}^*$,then

$$\mu^{t_j}(p) \geq \tilde{\delta}^*((q, \mu^{t_i}(q), x, p),$$

$\forall q \in Q_{(act)}(t_j), p \in Q$ and $x \in \sum$. As μ is a switching, then

$$\mu^{t_j}(p) \geq \tilde{\delta}^*((q, \mu^{t_i}(q), x, p),$$
$$= \tilde{\delta}^*((p, \mu^{t_i}(p), x, q),$$
$$= \mu^{t_j}(q)$$

As μ is a commutative, then $x = uv$

$$\mu^{t_j}(p) \geq \tilde{\delta}^*((q, \mu^{t_i}(q), x, p),$$
$$= \tilde{\delta}^*((q, \mu^{t_i}(q), uv, p),$$
$$= \vee\{\tilde{\delta}^*((q, \mu^{t_i}(q), u, r) \wedge \tilde{\delta}^*\left((r, \mu^{t_{i+1}}(r)), v, p\right) | r \in Q_{1(act)}(t_{i+1})\big)\}$$
$$= \vee\{\tilde{\delta}^*((r, \mu^{t_i}(r), u, q) \wedge \tilde{\delta}^*\left((p, \mu^{t_{i+1}}(p)), v, r\right) | r \in Q_{1(act)}(t_{i+1})\big)\}$$
$$= \vee\{\tilde{\delta}^*((p, \mu^{t_{i+1}}(p), v, r) \wedge \tilde{\delta}^*\left((r, \mu^{t_i}(r)), u, q\right) | r \in Q_{1(act)}(t_{i+1})\big)\}$$
$$= \tilde{\delta}^*((p, \mu^{t_{i+1}}(p), vu, q),$$
$$= \tilde{\delta}^*((p, \mu^{t_{i+1}}(p), uv, q),$$
$$= \tilde{\delta}^*((p, \mu^{t_{i+1}}(p), x, q),$$
$$\geq \mu^{t_j}(q)$$

According to the definition 5.4, μ is a strong switchboard subsystem if $\tilde{\delta}^*((q, \mu^{t_i}(q), x, p) > 0$, then for $q, p \in Q$ and $x \in \sum, \mu^{t_j}(p) \geq \mu^{t_j}(q)$. Since $\mu^{t_j}(p) \geq \mu^{t_j}(q)$, hence, μ is a strong switchboard subsystem of $\tilde{F}^*$.

Theorem 5.7:

Let $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a GFSA and let μ be a fuzzy subset of Q. If μ is a strong switchboard subsystem of $\tilde{F}^*$, then μ is a switchboard subsystem of $\tilde{F}^*$.

Proof:

Let $q, p \in Q$. Since μ is a strong switchboard subsystem of $\tilde{F}^*$, and μ is switching, $\forall x \in \sum$,

$$\mu^{t_j}(p) \geq \mu^{t_j}(q)$$

$$\geq \tilde{\delta}^*((p, \mu^{t_i}(p), x, q)(\text{since } \tilde{\delta}^*((q, \mu^{t_i}(q), x, p) > 0 \forall x \in \Sigma,)$$

$$\geq \tilde{\delta}^*((q, \mu^{t_i}(q), x, p),$$

It is clear that μ is commutative. Thus, μ is a switchboard subsystem of $\tilde{F}^*$.

Theorem 5.8:

Let $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$ be a GFSA. Let μ_1 and μ_2 be the switchboard subsystem of $\tilde{F}^*$. Then the following hold.

i. $\mu_1 \wedge \mu_2$ is a switchboard subsystem of $\tilde{F}^*$.
ii. $\mu_1 \vee \mu_2$ is a switchboard subsystem of $\tilde{F}^*$.
iii. $\mu_1 \wedge \mu_2$ is a strong switchboard subsystem of $\tilde{F}^*$.
iv. $\mu_1 \vee \mu_2$ is a strong switchboard subsystem of $\tilde{F}^*$.

Proof:

The proofs i, ii, iii, and iv are straightforward.

Definition 5.5:

Let $\tilde{F}^*{}_1 = (Q_1, \Sigma, \tilde{R}_1, Z, \tilde{\delta}^*{}_1, \omega, F_1, F_2)$ and $\tilde{F}^*{}_2 = (Q_2, \Sigma, \tilde{R}_2, Z, \tilde{\delta}^*{}_2, \omega, F_1, F_2)$ be a GFA. let $f: Q_1 \to Q_2$ and $g: \sum \to \sum$ be a mappings. A pair (f, g) is called a switching homomorphism if

$$\tilde{\delta}^*{}_2\left(\left(f(q), f(\mu^{t_j})(f(q))\right), g(x), f(p)\right) = \bigvee_{s,t \in Q_1} \{\tilde{\delta}^*{}_1\left((s, \mu^{t_j}(s)), x, t\right) | f(s)$$

$$= f(q), f(t) = f(p)\}$$

For $q, p \in Q_1$ and $x \in \sum$.

Definition 5.6:

Let $\tilde{F}^*{}_1 = (Q_1, \Sigma, \tilde{R}_1, Z, \tilde{\delta}^*{}_1, \omega, F_1, F_2)$ and $\tilde{F}^*{}_2 = (Q_2, \Sigma, \tilde{R}_2, Z, \tilde{\delta}^*{}_2, \omega, F_1, F_2)$ be a GFA. Let μ be a fuzzy subset of Q_1. Define the fuzzy subset $f(\mu)$ of Q_2 by

$$f(\mu)(q_2) = \begin{cases} \bigvee_{q_1 \in Q_1} \{\mu(q_1) | f(q_1) = q_2\} & if\ f^{-1}(q_2) \neq \emptyset \\ 0 & if\ f^{-1}(q_2) = \emptyset \end{cases}$$

$\forall q_2 \in Q_{2.}$

Theorem 5.9:

Let $\tilde{F}^*{}_1 = (Q_1, \Sigma, \tilde{R}_1, Z, \tilde{\delta}^*{}_1, \omega, F_1, F_2)$ be a GFSA and let μ be a fuzzy subset of Q, then μ is a (strong) switchboard subsystem of $\tilde{F}^*$. Let $\tilde{F}^*{}_2 = (Q_2, \Sigma, \tilde{R}_2, Z, \tilde{\delta}^*{}_2, \omega, F_1, F_2)$ be a GFFA and $(f, g): \tilde{F}^*{}_1 \to \tilde{F}^*{}_2$ be an onto switching homomorphism. The following hold:

i. $\tilde{F}^*{}_2$ is a GFSA,

ii. $f(\mu)$ is a switchboard subsystem of $\tilde{F}^*{}_2$, and

iii. $f(\mu)$ is a strong switchboard subsystem of $\tilde{F}^*{}_2$.

Proof:

Firstly, $\tilde{F}^*{}_2$ needs to be proven as a GFSA. Now, as for all $p_2, q_2 \in Q_2$. Since $f: Q_1 \to Q_2$ is onto mapping, $\exists p_1, q_1 \in Q_1$ such that $f(p_1) = p_2$ and $f(q_1) = q_2$. Besides, for all $u, v \in \sum$ needs to be considered. As $g: \sum \to \sum$ is onto mapping defined by $g(u) = u$ and $g(v) = v$.

Since $\tilde{F}^*{}_1$ is commutative, then

$$\tilde{\delta}^*{}_2\big((q_2, \mu^{t_j}(q_2)), uv, p_2\big) = \tilde{\delta}^*{}_2\Big(\Big(f(q_1), f(\mu^{t_j})\big(f(q_1)\big)\Big), g(u)g(v), f(p_1)\Big)$$

$$= \tilde{\delta}^*{}_2\Big(\Big(f(q_1), f(\mu^{t_j})\big(f(q_1)\big)\Big), g(uv), f(p_1)\Big)$$

$$= \vee \left\{\tilde{\delta}^*{}_1\Big(\big(s_1, \mu^{t_j}(s_1)\big), uv, t_1\Big) \,|\, s_1, t_1 \in Q_1, f(s_1) = f(q_1), f(t_1) = f(p_1)\right\}$$

$$= \vee \left\{\tilde{\delta}^*{}_1\Big(\big(s_1, \mu^{t_j}(s_1)\big) vu, t_1\Big) \,|\, s_1, t_1 \in Q_1, f(s_1) = f(q_1), f(t_1) = f(p_1)\right\}$$

$$= \tilde{\delta}^*{}_2\Big(\Big(f(q_1), f(\mu^{t_j})\big(f(q_1)\big)\Big), g(vu), f(p_1)\Big)$$

$$= \tilde{\delta}^*{}_2\Big(\Big(f(q_1), f(\mu^{t_j})\big(f(q_1)\big)\Big), g(v)g(u), f(p_1)\Big)$$

$$= \tilde{\delta}^*{}_2\big((q_2, \mu^{t_j}(q_2)), vu, p_2\big)$$

Hence $\tilde{F}^*{}_2$ is commutative.

Since $\tilde{F}^*{}_1$ is switching, then

$$\tilde{\delta}^*{}_2\big((q_2, \mu^{t_j}(q_2)), u, p_2\big) = \tilde{\delta}^*{}_2\Big(\Big(f(q_1), f(\mu^{t_j})\big(f(q_1)\big)\Big), g(u), f(p_1)\Big)$$

$$=\vee \{\tilde{\delta}^*{}_1\left(\left(s_1, \mu^{t_j}(s_1)\right), u, t_1\right) | s_1, t_1 \in Q_1, f(s_1) = f(q_1), f(t_1) = f(p_1)\}$$

$$=\vee \{\tilde{\delta}^*{}_1\left(\left(t_1, \mu^{t_j}(t_1)\right), u, s_1\right) | s_1, t_1 \in Q_1, f(s_1) = f(q_1), f(t_1) = f(p_1)\}$$

$$= \tilde{\delta}^*{}_2\left(\left(f(p_1), f(\mu^{t_j})\left(f(p_1)\right)\right), g(u), f(q_1)\right)$$

$$= \tilde{\delta}^*{}_2\left((p_2, \mu^{t_j}(p_2)), u, q_2\right)$$

Hence $\tilde{F}^*{}_2$ is switching. Thus $\tilde{F}^*{}_2$ is a GFSA.

Now, $f(\mu)$ needs to be proven as a switchboard subsystem of $\tilde{F}^*$. Let $p_2, q_2 \in Q_2$ and $u \in \sum$. Then

$$\tilde{\delta}^*{}_2\left((q_2, f(\mu^{t_j})(q_2)), u, p_2\right) = F_1(f(\mu^{t_j})(q_2), \tilde{\delta}^*{}_2(p_2, u, q_2))$$

$$= F_1(\bigvee_{q_1 \in Q_1} \{\mu^{t_j}(q_1) | f(q_1) = q_2\}, \tilde{\delta}^*{}_2(p_2, u, q_2))$$

$$=\vee_{q_1 \in Q_1}\{F_1\left(\mu^{t_j}(q_1), \tilde{\delta}^*{}_2(p_2, u, q_2)\right) | f(q_1) = q_2\}$$

Let $p_1, q_1 \in Q_1$ be such that $f(p_1) = p_2$ and $f(q_1) = q_3$. Then

$$F_1\left(\mu^{t_j}(q_1), \tilde{\delta}^*{}_2(p_2, u, q_2)\right) = F_1\left(\mu^{t_j}(q_1), \tilde{\delta}^*{}_2(f(q_2), u, f(p_2))\right)$$

$$= F_1(\mu^{t_j}(q_1), \left(\vee_{r_1 \in Q_1}\{\tilde{\delta}^*{}_1(q_1, u, r_1) \middle| f(r_1) = f(p_1) = p_2\}\right))$$

$$= \bigvee_{r_1 \in Q_1}\{F_1\left(\mu^{t_j}(q_1), \tilde{\delta}^*{}_1(q_1, u, r_1)\right) | f(r_1) = f(p_1) = p_2\}$$

$$= \bigvee_{r_1 \in Q_1}\{\left(\tilde{\delta}^*{}_1(q_1, \mu^{t_j}(q_1), u, r_1)\right) | f(r_1) = p_2\}$$

$$\leq f(\mu^{t_j})(p_1)$$

Therefore

$$\tilde{\delta}^*{}_2(q_2, f(\mu^{t_j})(q_2)), u, p_2) \leq \bigvee_{p_1 \in Q_1}\{f(\mu^{t_j})(p_2) | f(q_1) = q_2\}$$

$$= f(\mu^{t_j})(p_2)$$

Hence, $f(\mu)$ is a switchboard subsystem of $\tilde{F}^*{}_2$.

Since μ is a strong switchboard subsystem of $\tilde{F}^*$, then $\mu^{t_j}(r) \geq \mu^{t_j}(s)$ for all $r, s \in Q_1$. Assume that for $p_1, q_1 \in Q_1$, then

$$\bigvee_{r \in Q_1} \{\mu^{t_j}(r) | f(r) = f(q_1)\} \geq \bigvee_{s \in Q_1} \{\mu^{t_j}(s) | f(s) = f(p_1)\}$$

Let $\tilde{\delta}^*{}_2\left(\left(f(q_1), f(\mu^{t_j})(f(q_1))\right), u, f(p_1)\right) > 0$ for $u \in \Sigma$. Since

$$\tilde{\delta}^*{}_2\left(\left(f(q_1), f(\mu^{t_j})(f(q_1))\right), u, f(p_1)\right)$$

$$= \bigvee_{s,r \in Q_1} \{\tilde{\delta}^*{}_1((r, \mu^{t_j}(r), u, s \big| f(r) = f(q_1), f(s) = f(p_1)\},$$

There $\exists\, s, r \in Q_1$ such that $\tilde{\delta}^*{}_1((r, \mu^{t_j}(r), u, s) > 0$. Since μ is a strong switchboard subsystem, then

$$f(\mu^{t_j})\big(f(q_1)\big) = \bigvee_{r \in Q_1} \{\mu^{t_j}(r) | f(r) = f(q_1)\}$$

$$\geq \bigvee_{s \in Q_1} \{\mu^{t_j}(s) | f(s) = f(p_1)\}$$

$$= f(\mu^{t_j})\big(f(p_1)\big)$$

Thus, $f(\mu)$ is a strong switchboard subsystem of $\tilde{F}^*{}_2$.

5.4 Application of General Fuzzy Switchboard Automata

In this section, the applications of General Fuzzy Switchboard Automata, such as washing machines and rice cookers are studied. Switchboard properties have been applied to the machine because the switchboard can act as a control device and be able to communicate between one subsystem to another subsystem. Therefore, it is necessary to incorporate the idea of the switchboard properties to execute the machine. The calculations are stated below.

5.4.1 Washing machine

Figure 5.4 presents a General Fuzzy Switchboard Automata (GFSA) that describes the behavior of a washing machine which is influenced by the weight of the cloths in order to choose the suitable timer for the whole process. Once the state in that machine receives equal or more than 0.5 of membership value, it will go to the next process (state). Meanwhile, if the machine receives less than 0.5 of membership value, it means the timer should increase by 5 minutes. However, this condition occurs at rinse state only. Consider that the cloths have different materials, thickness and weight. Thus, the time for rinse is different.

In order to incorporate the switchboard property into the General Fuzzy Automata, two conditions have to be fulfilled which are the commutative and switching properties. Therefore, this system needs to check whether it is GFSA or GFA. Firstly, we need to check the commutative property of this system. From the Definition 5.1, if $\rho(q,\mu^{t_{i-1}}(q),uv,p)=\rho(q,\mu^{t_{i-1}}(q),vu,p)$, where $\forall q,p\in Q$, $u,v\in S, i\geq 1$, then it is called as commutative.

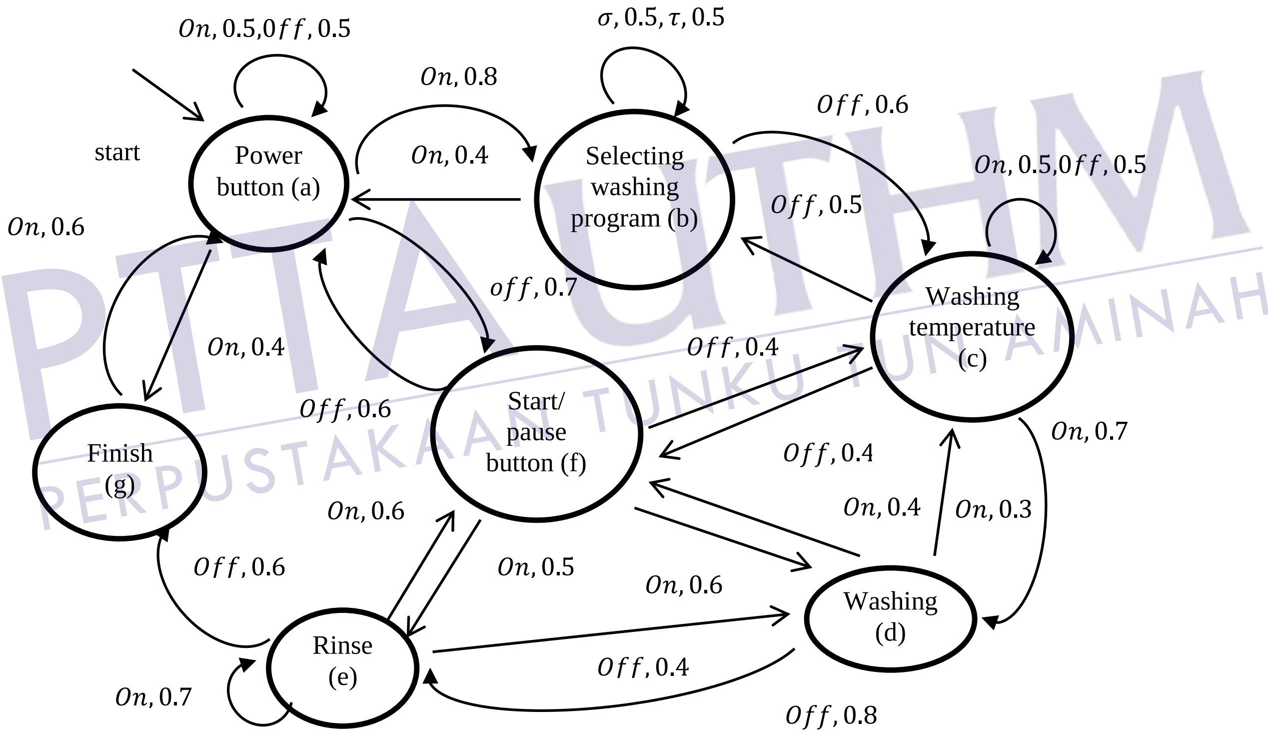

Figure 5.4: Application of a washing machine

Let $a,b,c,d,e,f,g\in Q$, and $on, off, \sigma,\tau\in\Sigma$.

Denote that a,b,c,d,e,f,g are the states of the system, while *on*, *off*, σ,τ are the input symbols with the membership values. String x represents the situation that will occur from the state. For instance, from the initial state (power button) to f (start/pause button), if the power is *off*, it can go to the f state. The meaning of input symbol *off* is that the state is still under process, meanwhile *on* means the state is already finished and does not operate in that state. At f state, to continue for the next state, if the input symbol is *on*, it will go to the *d* state (washing). Next, if there is something wrong with the machine, such as the water level is not suitable regarding the weight of the cloths and so on, it will go to the f state (*on*). However, assume that there is nothing wrong at the *d* state, then from the *d* state it will go to the *e* state (rinse). Assume, at the *e* state, the membership value is still greater than 0.5, means that the timer for rinse must be increased by 5 minutes. Thus at *e* state, the input symbol is on. Calculation below is for GFSA according to the situation given.

String $x = on, off, on, off$

$$\mu^{t_0}(a) = 1, \mu^{t_1}(b) = \tilde{\delta}\left((a, \mu^{t_0}(a)), on, b\right) = F_1\left(\mu^{t_0}(a), \delta(a, on, b)\right) = F_1(1{,}0.8)$$
$$= 0.8$$

$$\mu^{t_2}(c) = \tilde{\delta}\left((b, \mu^{t_1}(b)), off, c\right) = F_1\left(\mu^{t_1}(b), \delta(b, off, c)\right) = F_1(0.8{,}0.6) = 0.6$$

$$\mu^{t_3}(d) = \tilde{\delta}\left((c, \mu^{t_2}(c)), on, d\right) = F_1\left(\mu^{t_2}(c), \delta(c, on, d)\right) = F_1(0.6{,}0.7) = 0.6$$

$$\mu^{t_4}(e) = \tilde{\delta}\left((d, \mu^{t_3}(d)), off, e\right) = F_1\left(\mu^{t_3}(d), \delta(d, off, e)\right) = F_1(0.6{,}0.8) = 0.6$$

$$\tilde{\delta}^*\left((a, \mu^{t_0}(a)), on, b\right) = 0.8$$

$$\tilde{\delta}^*\left((a, \mu^{t_0}(a)), onoff, c\right) = 0.8 \wedge 0.6 = 0.6$$

$$\tilde{\delta}^*\left((a, \mu^{t_0}(a)), onoffon, d\right) = 0.8 \wedge 0.6 \wedge 0.6 = 0.6$$

$$\tilde{\delta}^*\left((a, \mu^{t_0}(a)), onoffonoff, e\right) = 0.8 \wedge 0.6 \wedge 0.6 \wedge 0.6 = 0.6$$

According to the commutative rule, if $\rho(q, \mu^{t_{i-1}}(q), uv, p) = \rho(q, \mu^{t_{i-1}}(q), vu, p)$, where $\forall q, p \in Q, u, v \in S, i \geq 1$. Since, the string $x = on, ff, on, off$, then the other side must be vice versa. Next check the calculation if the string $x = off, on, off, on$.

string $x = off, on, off, on$

$$\mu^{t_0}(a) = 1, \mu^{t_1}(f) = \tilde{\delta}\left((a, \mu^{t_0}(a)), off, f\right) = F_1\left(\mu^{t_0}(a), \delta(a, off, f)\right)$$
$$= F_1(1{,}0.7) = 0.7$$

$$\mu^{t_2}(d) = \tilde{\delta}\left((f, \mu^{t_1}(f)), on, d\right) = F_1\left(\mu^{t_1}(f), \delta(f, on, d)\right) = F_1(0.7{,}0.6) = 0.6$$

$$\mu^{t_3}(e) = \tilde{\delta}\left((d, \mu^{t_2}(d)), off, e\right) = F_1\left(\mu^{t_2}(d), \delta(d, off, e)\right) = F_1(0.6{,}0.8) = 0.6$$

$$\mu^{t_4}(e) = \tilde{\delta}\left((e, \mu^{t_3}(e)), on, e\right) = F_1\left(\mu^{t_3}(e), \delta(e, on, e)\right) = F_1(0.6{,}0.7) = 0.6$$

$$\tilde{\delta}^*\left((a, \mu^{t_0}(a)), off, f\right) = 0.7$$

$$\tilde{\delta}^*\left((a\ , \mu^{t_0}(a)), offon, d\right) = 0.7 \wedge 0.6 = 0.6$$

$$\tilde{\delta}^*\left((a, \mu^{t_0}(a)), offonoff, e\right) = 0.7 \wedge 0.6 \wedge 0.6 = 0.6$$

$$\tilde{\delta}^*\left((a, \mu^{t_0}(a)), offonoffon, e\right) = 0.7 \wedge 0.6 \wedge 0.6 \wedge 0.6 = 0.6$$

The tables below show the operation of fuzzy automaton upon input string $(onoff)^2$and $(offon)^2$ for F_1 and F_2.

Table 5.5: Active states and their membership values of $onoffonoff$

Time	t_0	t_1	t_2	t_3	t_4
input	Λ	on	off	on	off
$Q_{act}(t_i)$	a	b	c	d	e
Membership value	1.0	0.8	0.6	0.6	0.6

Table 5.6 : Active states and their membership values of $offonoffon$

Time	t_0	t_1	t_2	t_3	t_4
input	Λ	off	on	$offf$	on
$Q_{act}(t_i)$	a	f	d	e	e
Membership value	1.0	0.7	0.6	0.6	0.6

Since, $\rho(q, \mu^{t_{i-1}}(q), onoffonoff, p) = \rho(q, \mu^{t_{i-1}}(q), offonoffon, p)$, thus, the system is commutative. Next, check for the switching properties. If $\rho(q, \mu^{t_{i-1}}(q), u, p) = \rho(p, \mu^{t_{i-1}}(p), u, q)$, where $\forall q, p \in Q,\ u, v \in S, i \geq 1$, then the system is switching. Here, from the diagram above, it shows that every state is switching. For instance, $\rho(a, \mu^{t_{i-1}}(a), on, b) = \rho(b, \mu^{t_{i-1}}(b), on, a)$. Thus, the system is switching. Therefore, the system is GFSA since it fulfilled both conditions switching and commutative.

5.4.2 Rice cooker

Figure 5.5 represents the application of a rice cooker with GFSA. The circles represent the state of the system while the arrows from one state to another state represent the transition. a, b, c are the types of functions, such as porridge, steam, and grilled, meanwhile ρ represents the time of the types of functions, where every type of function has a specific time. For instance, if the user selects the menu is porridge, thus the time is 30 minutes.

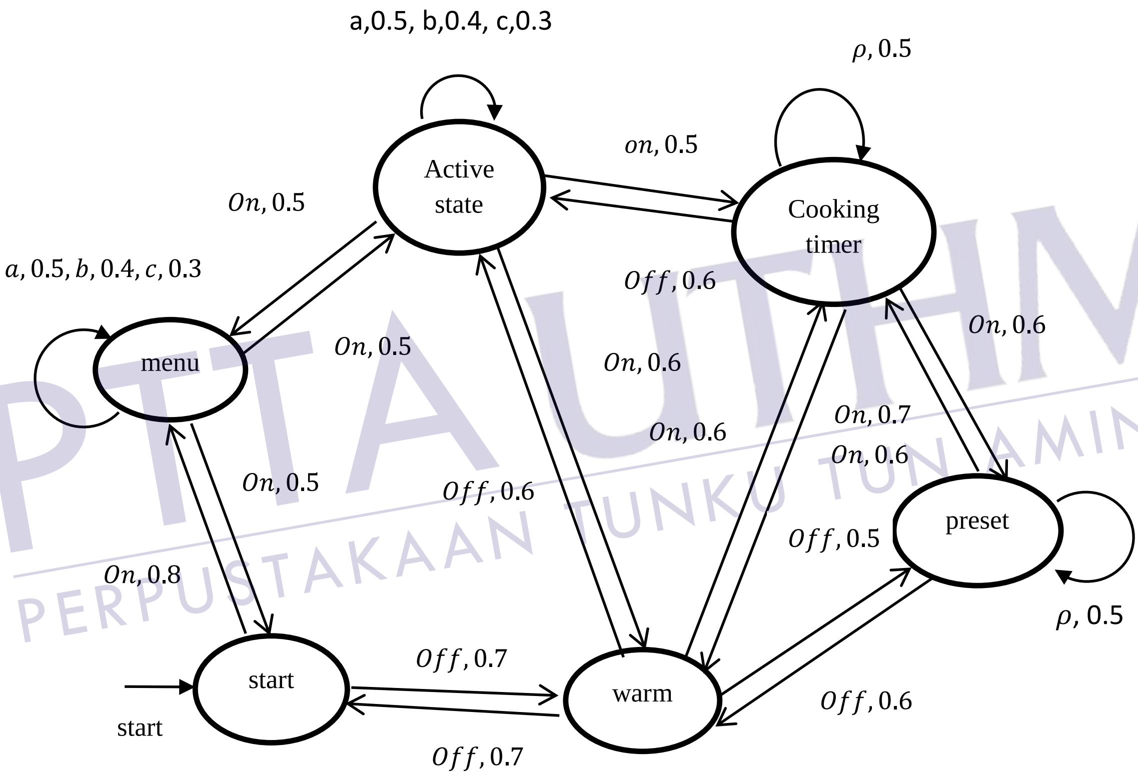

Figure 5.5: The simple system of a rice cooker

Let $start, menu, active\ state, cooking\ timer, preset, warm \in Q$,and $on, off, a, b, c, \rho \in \Sigma$

Basically, check the commutative and switching state machine of this system.

String $x = on, on, off, off$ then

$$\mu^{t_0}(start) = 1, \mu^{t_1}(menu) = \tilde{\delta}\left(\left(start, \mu^{t_0}(start)\right), on, menu\right)$$

$$= F_1\left(\mu^{t_0}(start), \delta(start, on, menu)\right) = F_1(1,0.8) = 0.8$$

$$\mu^{t_2}(active\ state) = \tilde{\delta}\left((menu, \mu^{t_1}(menu)), on, active\ state\right)$$

$$= F_1\big(\mu^{t_1}(menu), \delta(menu, on, active\ state)\big) = F_1(0.8{,}0.5) = 0.5$$

$$\mu^{t_3}(warm) = \tilde{\delta}\left((active\ state, \mu^{t_2}(active\ state)), off, warm\right)$$

$$= F_1\big(\mu^{t_2}(active\ state), \delta(active\ state, off, warm)\big)$$

$$= F_1(0.5{,}0.6) = 0.5$$

$$\mu^{t_4}(start) = \tilde{\delta}\left((warm, \mu^{t_3}(warm)), off, start\right)$$

$$= F_1\big(\mu^{t_3}(warm), \delta(warm, off, start)\big) = F_1(0.5{,}0.7) = 0.5$$

$$\tilde{\delta}^*\left((start, \mu^{t_0}(start)), on, menu\right) = 0.8$$

$$\tilde{\delta}^*\left((start, \mu^{t_0}(start)), onon, active\ state\right) = 0.8 \wedge 0.5 = 0.5$$

$$\tilde{\delta}^*\left((start, \mu^{t_0}(start)), ononoff, warm\right) = 0.8 \wedge 0.5 \wedge 0.5 = 0.5$$

$$\tilde{\delta}^*\left((start, \mu^{t_0}(start)), ononoffoff, start\right) 0.8 \wedge 0.5 \wedge 0.5 \wedge 0.5 = 0.5$$

If choose input string $x = off, off, on, on$

$$\mu^{t_0}(start) = 1, \mu^{t_1}(warm) = \tilde{\delta}\left((start, \mu^{t_0}(start)), off, warm\right)$$

$$= F_1\big(\mu^{t_0}(start), \delta(start, off, warm)\big) = F_1(1{,}0.7) = 0.7$$

$$\mu^{t_2}(active\ state) = \tilde{\delta}\left((warm, \mu^{t_1}(warm)), off, active\ state\right)$$

$$= F_1\big(\mu^{t_1}(warm), \delta(warm, off, active\ state)\big) = F_1(0.7{,}0.6) = 0.6$$

$$\mu^{t_3}(menu) = \tilde{\delta}\left((active\ state, \mu^{t_2}(active\ state)), on, menu\right)$$

$$= F_1\big(\mu^{t_2}(active\ state), \delta(active\ state, on, menu)\big) = F_1(0.6{,}0.5)$$

$$= 0.5$$

$$\mu^{t_4}(start) = \tilde{\delta}\left((menu, \mu^{t_3}(menu)), on, start\right)$$

$$= F_1\big(\mu^{t_3}(menu), \delta(menu, on, start)\big) = F_1(0.5{,}0.8) = 0.5$$

$$\tilde{\delta}^*\left((start, \mu^{t_0}(start)), off, warm\right) = 0.7$$

$$\tilde{\delta}^*\left((start, \mu^{t_0}(start)), offoff, active\ state\right) = 0.7 \wedge 0.6 = 0.6$$

$$\tilde{\delta}^*\left((start, \mu^{t_0}(start)), offoffon, menu\right) = 0.7 \wedge 0.6 \wedge 0.5 = 0.5$$

$$\tilde{\delta}^*\left((start, \mu^{t_0}(start)), offoffonon, start\right) 0.7 \wedge 0.6 \wedge 0.5 \wedge 0.5 = 0.5$$

The tables below show the operation of fuzzy automaton upon input string $ononoffoff$ and $offoffonon$ for F_1 and F_2.

Table 5.7: Active states and their membership values of $ononoffoff$ for a rice cooker

time	t_0	t_1	t_2	t_3	t_4
input	Λ	on	on	off	off
$Q_{act}(t_i)$	$start$	$menu$	$active\ state$	$warm$	$start$
Membership value	1.0	0.8	0.5	0.5	0.5

Table 5.8 : Active states and their membership values of $offoffonon$ for a rice cooker

time	t_0	t_1	t_2	t_3	t_4
input	Λ	off	off	on	on
$Q_{act}(t_i)$	$start$	$warm$	$active\ state$	$menu$	$st\square rt$
Membership value	1.0	0.7	0.6	0.5	0.5

Since, $\rho(q, \mu^{t_{i-1}}(q), ononoffoff, p) = \rho(q, \mu^{t_{i-1}}(q), offoffonon, p)$, thus the system is commutative.

For the switching properties, since $\rho(q, \mu^{t_{i-1}}(q), u, p) = \rho(p, \mu^{t_{i-1}}(p), u, q)$, where $\forall q, p \in Q$, $u, v \in S, i \geq 1$, then the system is switching. As example from the diagram above,

$$\rho(start, \mu^{t_{i-1}}(start), on, menu) = \rho(menu, \mu^{t_{i-1}}(menu), on, start).$$

Thus, the system is switching. Therefore, the system is GFSA since it fulfilled both conditions switching and commutative.

5.5 Summary

By incorporating the switchboard into the General Fuzzy Automata (GFA), the definition and the notion of General Fuzzy Switchboard Automata (GFSA) have been introduced. The examples show that several states consist of multi-membership values. Thus, by applying GFSA, the calculations of the membership values of the states are shown. The system has to be checked whether it follows the properties of a switchboard automata. The subsystem is a self-contained system that is part of a larger system. The system can contain several subsystems depending on the system itself. All the operations in the system are independent of each other. The switchboard subsystem can interact with the subsystems automatically when the system starts operating. The switchboard subsystem and switchboard strong subsystem are introduced in the environment of GFSA. Real-life applications of GFSA, such as washing machines and rice cookers are illustrated. The importance of GFSA in the real-life application is, it can enhance the system to operate well and automatically. If sudden failure happens, it can communicate between the subsystems and decide for the human being. By extending the algebraic properties of GFSA, the General Fuzzy Switchboard Transformation Semigroup is examined in Chapter 6.

CHAPTER 6

TRANSFORMATION SEMIGROUP IN GENERAL FUZZY SWITCHBOARD AUTOMATA

6.1 Introduction

In this chapter, the concept of General Fuzzy Switchboard Transformation Semigroup (GFSTS) is introduced by the combination of fuzzy finite transformation semigroup, switchboard properties and general fuzzy automata. Some related definitions and properties are established. The covering and some products, such as direct and cascade of GFSTS are also studied.

6.2 Fuzzy finite transformation semigroup

Transformation semigroup is a pair (Q, S) where Q is a finite nonempty set and S is a finite semigroup with an action ρ of S on Q, $\rho: Q \times S \times Q$. The action means that the element of a semigroup is acting as a transformation of the set by using operation that associates two elements of the semigroup. Basically, in the concept of computation and science, finite transformation semigroup is the notion of change from one state to another state in a system due to the internal process of various time scales or due to external manipulation. Transformation semigroup defined all the different ways (transition) set transformations that can be combined in time besides as a collection of the functions from a set to itself. By listing these semigroups, it is easier to explore the space of all possible finite computations since they have an enormous set of states.

Definition 6.0:

Let $TS = (Q, S, \rho)$ be fuzzy transformation semigroup.

i. TS is called switching if and only if

$$\rho(p, x, q) = \rho(q, x, p)$$

for $\forall p, q \in Q, \forall x \in S$.

ii. TS is called commutative if and only if

$$\rho(p, xy, q) = \rho(p, yx, q)$$

for $\forall p, q \in Q, \forall x, y \in S$.

If TS is switching and commutative, then TS is called a **Finite Switchboard Transformation Semigroup (FSTS)**.

Definition 6.1:

Let $TS_1 = (Q_1, S_1, \rho_1)$ and $TS_2 = (Q_2, S_2, \rho_2)$ be a switchboard transformation semigroup over L and X. A strong homomorphism from TS_1 and TS_2 is a pair (α, β) of mappings $\alpha: TS_1 \to TS_2$ and $\beta: X_1 \to X_2$ such that

$$\rho_1(p, x, q) \leq_L \rho_2(\alpha(p), \beta(x), \alpha(q))$$

for any $p, q \in Q_1$ and $x \in X_1$.

Lemma 6.0:

Let $TS_1 = (Q_1, S_1, \rho_1)$ be a commutative transformation semigroup and $TS_2 = (Q_2, S_2, \rho_2)$ be a transformation semigroup. Let $(\alpha, \beta): TS_1 \to TS_2$ be an onto strong homomorphism. Then, TS_2 is a commutative transformation semigroup.

Definition 6.2:

Let $TS_1 = (Q_1, S_1, \rho_1)$ and $TS_2 = (Q_2, S_2, \rho_2)$ be a transformation semigroup over L and X. A switching homomorphism from TS_1 and TS_2 is a pair (α, β) of mappings $\alpha: TS_1 \to TS_2$ and $\beta: X_1 \to X_2$ such that,

$$\rho(p, x, q) = \rho(q, x, p)$$

for $\forall p, q \in Q, \forall x \in S$.

6.3 General Fuzzy Switchboard Transformation Semigroup (GFSTS)

In this section, the definitions and the notion of GFSTS are introduced. The properties of GFSTS and the proving are shown.

Definition 6.3:

A fuzzy transformation semigroup (fts) is a triple $TS(\tilde{F}) = (Q, S(\tilde{F}), \rho)$ where Q is a finite nonempty set, $S(\tilde{F})$ is a finite semigroup of $\tilde{F}$, ρ is a fuzzy subset of $Q \times S(\tilde{F}) \times Q$.

Such that

i. $\rho(q, \mu^{t_i}(q), uv, p) = \vee_{r \in Q}\{\rho(q, \mu^{t_i}(q), u, r) \wedge \rho(r, \mu^{t_{i+1}}(r), v, p)\}$, for all $u, v \in S$, $q, p \in Q, i \geq 0$.

ii. If S contains the identity e, then $\rho(q, \mu^{t_i}(q), e, p) = 1$ if $q = p$ and $\rho(q, \mu^{t_i}(q), e, p) = 0$ if $q \neq p$, $\forall q, p \in Q, i \geq 0$.

If the property holds, then $TS(\tilde{F}) = (Q, S(\tilde{F}), \rho)$ is called **faithful**.

iii. Let $u, v \in S(\tilde{F})$, if $\rho(p, \mu^{t_i}(p), u) = \rho(p, \mu^{t_i}(p), v), \forall p \in Q, i \geq 0$ then $u = v$.

Definition 6.4:

Let $T = (Q, S(\tilde{F}), \rho)$ be a fuzzy transformation semigroup (fts). Then

i. T is commutative if it satisfied $\rho(q, \mu^{t_{i-1}}(q), uv, p) = \rho(q, \mu^{t_{i-1}}(q), vu, p)$, $\forall q, p \in Q, \forall u, v \in S(\tilde{F})$, $i \geq 1$.

ii. T is switching if it satisfied $\rho(q, \mu^{t_i}(q), uv, p) = \rho(p, \mu^{t_i}(p), uv, q)$, $\forall q, p \in Q, \forall u, v \in S(\tilde{F})$, $i \geq 0$.

If T satisfied both conditions which are commutative and switching, thus it is called as **General Fuzzy Switchboard Transformation Semigroup (GFSTS).**

Proposition 6.0:

Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy switchboard automata. Then T is a general fuzzy switchboard transformation semigroup.

Lemma 6.1:

$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ is a commutative GFSTS for every $i \geq 1$,

$\rho(q, \mu^{t_{i-1}}(q), uv, p) = \rho(q, \mu^{t_{i-1}}(q), vu, p),$ for all $q, p \in Q$, $u \in S(\tilde{F})$, $v \in S(\tilde{F})^*$.

Proof:

If $\tilde{F}^*$ is commutative, clearly T is also commutative. Denote $\rho(q, \mu^{t_{i-1}}(q), uv, p) = \rho(q, \mu^{t_{i-1}}(q), vu, p)$, where $\forall q, p \in Q$, $u, v \in S(\tilde{F})$, $i \geq 1$.

Let $v \in S(\tilde{F})^*$ and $|v| = n$. If $n = 0$, then $v = \lambda$. Hence,

$$
\begin{aligned}
\rho(q, \mu^{t_{i-1}}(q), uv, p) &= \rho(q, \mu^{t_{i-1}}(q), u\lambda, p) \\
&= \rho(q, \mu^{t_{i-1}}(q), u, p) \\
&= \rho(q, \mu^{t_{i-1}}(q), \lambda u, p) \\
&= \rho(q, \mu^{t_{i-1}}(q), vu, p)
\end{aligned}
$$

Now suppose the result is true for all $x \in S(\tilde{F})^*$, such that $|x| = n - 1, n > 0$. Let $u = xy$ where $y \in S(\tilde{F})$ and $x \in S(\tilde{F})^*$. Then

$$
\begin{aligned}
\rho^*(q, \mu^{t_{i-1}}(q), uv, p) &= \rho^*(q, \mu^{t_{i-1}}(q), xyv, p) \\
&= \bigvee_{r \in Q} \{\rho(q, \mu^{t_{i-1}}(q), x, r) \wedge \rho(r, \mu^{t_i}(r), yv, p)\} \\
&= \bigvee_{r \in Q} \{\rho(q, \mu^{t_{i-1}}(q), x, r) \wedge \rho(r, \mu^{t_i}(r), vy, p)\} \\
&= \rho^*(q, \mu^{t_{i-1}}(q), xvy, p) \\
&= \bigvee_{r \in Q} \{\rho(q, \mu^{t_{i-1}}(q), xv, r) \wedge \rho(r, \mu^{t_i}(r), y, p)\} \\
&= \bigvee_{r \in Q} \{\rho(q, \mu^{t_{i-1}}(q), vx, r) \wedge \rho(r, \mu^{t_i}(r), y, p)\} \\
&= \rho^*(q, \mu^{t_{i-1}}(q), vxy, p) \\
&= \rho^*(q, \mu^{t_{i-1}}(q), vu, p)
\end{aligned}
$$

The lemma 6.1 completes the proof and the result now follows by induction.

Lemma 6.2:

$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ is a switching GFSTS for every $i \geq 0$,

$\rho(q, \mu^{t_i}(q), uv, p) = \rho(p, \mu^{t_i}(p), uv, q)$, for all $q, p \in Q$, $u \in S(\tilde{F})$, $v \in S(\tilde{F})$ *.

Proof:

Let $u, v \in S(\tilde{F})$ * and $|uv| = n$. If $n = 0$, then $uv = \lambda$.

$$\rho(q, \mu^{t_i}(q), uv, p) = \bigvee_{r \in Q} \{\rho(q, \mu^{t_i}(q), u, r) \wedge \rho(r, \mu^{t_{i+1}}(r), v, p)\}$$

$$= \bigvee_{r \in Q} \{\rho(r, \mu^{t_i}(r), u, q) \wedge \rho(p, \mu^{t_{i+1}}(p), v, r)\}$$

$$= \bigvee_{r \in Q} \{(p, \mu^{t_i}(p), v, r) \wedge \rho(r, \mu^{t_{i+1}}(r), u, q\}$$

$$= \rho^*(p, \mu^{t_i}(p), vu, q)$$

$$= \rho^*(p, \mu^{t_i}(p), vu, q)$$

Hence, the result is true and follows by induction.

6.3.1 Covering

Sato and Kuroki (2002) found that the concept of covering is effective in studies of product besides it is useful for studies on state machines and transformation semigroup.

Definition 6.5:

Let $T_k = (Q_k, S(\tilde{F})_k, \rho_k)$ be GFSTS, $k = 1,2$. Let η be a function of Q_2 onto Q_1 and let ξ be a function of $S(\tilde{F})_1$ into $S(\tilde{F})_2$. Extend ξ to a function ξ^* of $S(\tilde{F})_1^*$ into $S(\tilde{F})_2^*$ by $\xi^*(\Lambda) = \Lambda$ and $\forall s \in S(\tilde{F})_1^*$, $\xi^*(s) = \xi(s_1)\xi(s_2) \cdots \xi(s_n)$ where $s = s_1 s_2 \cdots s_n$ and $s_k \in S(\tilde{F})_1$, $k = 1,2, \cdots, n$. Then (η, ξ) is called covering of T_1 by T_2, written $T_1 \leq T_2$ if and only if $\forall q_1 \in Q_2, p_1 \in Q_1$ and $s \in S(\tilde{F})_1^*$ where $i \geq 0$.

$$\rho_1^*(\eta(q_1, \mu^{t_i}(q_1), s, p_1) = \vee\{\rho_2^*(q_1, \mu^{t_i}(q_1), \xi^*(s), q_2) | \eta(q_2) = q_1, q_2 \in Q_2\}$$

Clearly, (η,ξ) is a covering of T_1 by T_2 if and only if $\forall q_1 \in Q_2, p_1 \in Q_1$ and $s \in S(\tilde{F})_1^*$ where $i \geq 0, \rho_1^*(\eta(q_1,\mu^{t_i}(q_1),s,p_1) \geq \rho_2^*(q_1,\mu^{t_i}(q_1),\xi^*(s),q_2) \forall q_2 \in Q_2$ such that $\eta(q_2) = p_1$ and $\exists q_2 \in Q_2$.

Example 6.3.1.1:

Let $T_1 = (Q_1, S(\tilde{F})_1, \rho_1)$ and $T_2 = (Q_2, S(\tilde{F})_2, \rho_2)$ be General Fuzzy Switchboard Transformation Semigroup (GFSTS) such that $Q_1 = \{p_1, p_2\}, S(\tilde{F})_1 = \{s_1, t_1\}$, $Q_2 = \{q_1, q_2, q_3\}, S(\tilde{F})_2 = \{s_2, t_2\}$ where $i \geq 0$ and ρ_1 and ρ_2 are defined as follows:

$$\rho_1(p_1, \mu^{t_i}(p_1), s_1, p_1) = 0.3$$

$$\rho_1(p_1, \mu^{t_i}(p_1), t_1, p_2) = 0.6$$

$$\rho_1(p_2, \mu^{t_i}(p_2), s_1, p_2) = 0.3$$

$$\rho_1(p_2, \mu^{t_i}(p_2), t_1, p_1) = 0.6$$

$$\rho_2(q_1, \mu^{t_i}(q_1), s_2, q_3) = 0.3$$

$$\rho_2(q_1, \mu^{t_i}(q_1), t_2, q_2) = 0.6$$

$$\rho_2(q_2, \mu^{t_i}(q_2), s_2, q_2) = 0.3$$

$$\rho_2(q_2, \mu^{t_i}(q_2), t_2, q_3) = 0.6$$

$$\rho_2(q_3, \mu^{t_i}(q_3), s_2, q_1) = 0.3$$

$$\rho_2(q_3, \mu^{t_i}(q_3), t_2, q_2) = 0.25$$

And $\rho_1(p_1, \mu^{t_i}(p_1), s, p_2) = 0$ for all other $(p_1, s, p_2) \in Q_1 \times S(\tilde{F})_1 \times Q_1$ and $\rho_2(q_1, \mu^{t_i}(q_1), s, q_2) = 0$ for all $(q_1, s, q_2) \in Q_2 \times S(\tilde{F})_2 \times Q_2$. Define $\eta: Q_2 \to Q_1$ by $\eta(q_1) = \eta(q_3) = p_1$ and $\eta(q_2) = p_2$. Let ξ be the identity map on $S(\tilde{F})_1 \times S(\tilde{F})_2$. Now, for all $s \in S(\tilde{F})_1^* = S(\tilde{F})_2^*$.

$$\rho_1^*(p_1, \mu^{t_i}(p_1), s, p_1) = \rho_2^*(q_1, \mu^{t_i}(q_1), s, q_1) \vee \rho_2^*(q_1, \mu^{t_{i+1}}(q_1), s, q_3)$$

$$\rho_1^*(p_1, \mu^{t_i}(p_1), s, p_2) = \rho_2^*(q_1, \mu^{t_i}(q_1), s, q_2)$$

$$\rho_1^*(p_2, \mu^{t_i}(p_2), s, p_1) = \rho_2^*(q_1, \mu^{t_i}(q_1), s, q_2) \vee \rho_2^*(q_2, \mu^{t_{i+1}}(q_1), s, q_3)$$

$$\rho_1^*(p_2, \mu^{t_i}(p_2), s, p_2) = \rho_2^*(q_2, \mu^{t_i}(q_2), s, q_2)$$

Thus $((\eta, \xi)$ is a covering of T_1 by T_2.

6.3.2 Direct product of General Fuzzy Switchboard Transformation Semigroup

In this section, the definition of direct product of GFSTS is introduced. The properties and proving are shown in below.

Definition 6.6:

Let $T_1 = (Q_1, S(\tilde{F})_1, \rho_1)$ and $T_2 = (Q_2, S(\tilde{F})_2, \rho_2)$ be a general fuzzy switchboard transformation semigroup. $S(\tilde{F})$ is a finite semigroup of $\tilde{F}$, and f are function of $S(\tilde{F})$ into $S(\tilde{F})_1 \times S(\tilde{F})_2$ written as $f: S(\tilde{F}) \to S(\tilde{F})_1 \times S(\tilde{F})_2$ is a map. Write as $f(s) = (f_1(s), f_2(s))$ for any $s \in S(\tilde{F})$. If

$\rho_{1\mathrm{f}}\, \rho_2 : (Q_1 \times Q_2) \times S(\tilde{F}) \times (Q_1 \times Q_2) \to [0,1]$

Given as $\forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $\forall s \in S(\tilde{F})$ where L is any distributive lattice and $i \geq 0$.

$$\rho_{1f}\, \rho_2\big((p_1, \mu^{t_i}(p_1), p_2, \mu^{t_i}(p_2)), s, (q_1, q_2)\big)$$
$$= \rho_1(p_1, \mu^{t_i}(p_1), f_1(s), q_1) \wedge \rho_2(p_2, \mu^{t_i}(p_2), f_2(s), q_2)$$

Then, $(Q_1 \times Q_2), S(\tilde{F}), \rho_{1\mathrm{f}}\, \rho_2)$ is called the general direct product of T_1 and T_2 and denoted by $T_1 \wedge T_2$. Denoted $\rho_1 \wedge \rho_2$ given by $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $s \in S(\tilde{F})$ by

$$\rho_1 \wedge \rho_2(p_1, \mu^{t_i}(p_1), p_2, \mu^{t_i}(p_2), s, (q_1, q_2)$$
$$= \rho_1(p_1, \mu^{t_i}(p_1), s, q_1) \wedge \rho_2(p_2, \mu^{t_i}(p_2), s, q_2)$$

Proposition 6.2:

Let $T_1 = (Q_1, S(\tilde{F})_1, \rho_1)$ and $T_2 = (Q_2, S(\tilde{F})_2, \rho_2)$ be General Fuzzy Switchboard Transformation Semigroup (GFSTS). Then $T_1 \times T_2$ is a GFSTS if and only if T_1 and T_2 are GFSTS.

Proof:

$\forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $\forall (s_1, s_2) \in S(\tilde{F})_1 \times S(\tilde{F})_2$ where $i \geq 1$, then

$$\rho_1(p_1, \mu^{t_{i-1}}(p_1), s_1, q_1) \wedge \rho_2(p_2, \mu^{t_{i-1}}(p_2), s_2, q_2)$$
$$= \rho_1 \times \rho_2(p_1, \mu^{t_{i-1}}(p_1), p_2, \mu^{t_{i-1}}(p_2)), (s_1, s_2), (q_1, q_2)$$

since T_1 and T_2 are GFSTS, then

$$\rho_1(p_1, \mu^{t_{i-1}}(p_1), s_1, q_1) = \rho_1(q_1, \mu^{t_{i-1}}(q_1), s_1, p_1);$$

$$\rho_2(p_2, \mu^{t_{i-1}}(p_2), s_2, q_2) = \rho_2(q_2, \mu^{t_{i-1}}(q_2) s_2, p_2)$$

$\forall p_1, q_1 \in Q_1, p_2, q_2 \in Q_2$ and $s_1 \in S(\tilde{F})_1, s_2 \in S(\tilde{F})_2$ where $i \geq 0$.

$T_1 \times T_2$ is switching shown as below:

$$\rho_1 \times \rho_2\big((q_1, \mu^{t_i}(q_1), q_2, \mu^{t_i}(q_2)), (s_1, s_2), (p_1, p_2)\big)$$
$$= \rho_1(q_1, \mu^{t_i}(q_1), s_1, p_1) \wedge \rho_2(q_2, \mu^{t_i}(q_2), s_2, p_2)$$
$$= \rho_1(p_1, \mu^{t_i}(p_1), s_1, q_1) \wedge \rho_2(p_2, \mu^{t_i}(p_2), s_2, q_2)$$
$$= \rho_1 \times \rho_2((p_1, \mu^{t_i}(p_1), p_2, \mu^{t_i}(p_2)), (s_1, s_2), (q_1, q_2))$$

Hence, $T_1 \times T_2$ is switching.

Let $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$, $(s_1, s_2), (t_1, t_2) \in S(\tilde{F})_1 \times S(\tilde{F})_2$ where $i \geq 1$. Since T_1 and T_2 are commutative, then

$$\rho_1(p_1, \mu^{t_{i-1}}(p_1), s_1 t_1, q_1) = \rho_1(p_1 \mu^{t_{i-1}}(p_1), t_1 s_1, q_1);$$

$$\rho_2(p_2, \mu^{t_{i-1}}(p_2), s_2 t_2, q_2) = \rho_2(p_2, \mu^{t_{i-1}}(p_2), t_2 s_2, q_2)$$

Thus, $\rho_1 \times \rho_2\Big((p_1, \mu^{t_{i-1}}(p_1), p_2, \mu^{t_{i-1}}(p_2)), (s_1, s_2)(t_1, t_2), (q_1, q_2)\Big)$

$$= \rho_1 \times \rho_2\big((p_1, \mu^{t_{i-1}}(p_1), p_2, \mu^{t_{i-1}}(p_2)), (s_1 t_1, s_2 t_2), (q_1, q_2)\big)$$

$$= \rho_1(p_1\mu^{t_{i-1}}(p_1), s_1t_1, q_1) \wedge \rho_2(p_2, \mu^{t_{i-1}}(p_2), s_2t_2, q_2)$$

$$= \rho_1(p_1, \mu^{t_{i-1}}(p_1), t_1s_1, q_1) \wedge \rho_2(p_2, \mu^{t_{i-1}}(p_2), t_2s_2, q_2)$$

$$= \rho_1 \times \rho_2\left(\left(p_1, \mu^{t_{i-1}}(p_1), p_2, \mu^{t_{i-1}}(p_2)\right), (t_1s_1, t_2s_2), (q_1, q_2)\right)$$

$$= \rho_1 \times \rho_2\left(\left(p_1, \mu^{t_{i-1}}(p_1), p_2, \mu^{t_{i-1}}(p_2)\right), (t_1, t_2)(s_1, s_2), (q_1, q_2)\right)$$

Which means $T_1 \times T_2$ is commutative. Therefore, $T_1 \times T_2$ is a GFSTS.

Since $T_1 \times T_2$ is switching, $\forall p_1, q_1 \in Q_1, p_2, q_2 \in Q_2$ and $s_1 \in S(\tilde{F})_1, s_2 \in S(\tilde{F})_2$ where $i \geq 0$.

Then,

$$\rho_1(p_1, \mu^{t_i}(p_1), s_1, q_1) \wedge \rho_2(p_2, \mu^{t_i}(p_2), s_2, q_2)$$

$$= \rho_1 \times \rho_2((p_1, \mu^{t_i}(p_1), p_2, \mu^{t_i}(p_2)), (s_1, s_2), (q_1, q_2))$$

$$= \rho_1 \times \rho_2((q_1, \mu^{t_i}(q_1), q_2, \mu^{t_i}(q_2)), (s_1, s_2), (p_1, p_2))$$

$$= \rho_1(q_1, \mu^{t_i}(q_1), s_1, p_1) \wedge \rho_2(q_2, \mu^{t_i}(q_2), s_2, p_2)$$

Therefore, $(p_1, \mu^{t_i}(p_1), s_1, q_1) = \rho_1(q_1, \mu^{t_i}(q_1), s_1, p_1)$ and $\rho_2(p_2, \mu^{t_i}(p_2), s_2, q_2) = \rho_2(q_2, \mu^{t_i}(q_2), s_2, p_2)$ imply T_1 and T_2 are switching. T_1 and T_2 also commutative. Thus, T_1 and T_2 are GFSTS.

6.3.3 Cascade product in General Fuzzy Switchboard Transformation Semigroup

The definition of cascade product in GFSTS is introduced. Some properties and proving are represented.

Definition 6.7:

Let $T_1 = (Q_1, S(\tilde{F})_1, \rho_1)$ and $T_2 = (Q_2, S(\tilde{F})_2, \rho_2)$ be GFSTS. Define the restricted cascade product $T_1 \varpi T_2 = (Q_1 \times Q_2, S(\tilde{F})_2, \rho^{\varpi})$ of T_1 and T_2 with respect to mapping $\varpi: S(\tilde{F})_2 \to S(\tilde{F})_1$ as,

$$\rho^{\varpi}((p_1,\mu^{t_i}(p_1),p_2,\mu^{t_i}(p_2)),s_2,(q_1,q)$$
$$=\rho_1(p_1,\mu^{t_i}(p_1),\varpi(s_2),q_1)\wedge\rho_2(q_2,\mu^{t_i}(q_2),s_2,p_2),$$

Where $\rho^{\varpi}:(Q_1\times Q_2)\times S(\tilde{F})_2\times(Q_1\times Q_2)\to[0,1],\forall(p_1,p),(q_1,q_2)\in Q_1\times Q_2$ and $s_2\in S(\tilde{F})_2$ where $i\geq 0$.

Proposition 6.3:

Let $T_1=(Q_1,S(\tilde{F})_1,\rho_1)$ and $T_2=(Q_2,S(\tilde{F})_2,\rho_2)$ be GFSTS. Then, there exists $\omega: Q_2\times S(\tilde{F})_2\to S(\tilde{F})_1$ $\forall\varpi: S(\tilde{F})_2\to S(\tilde{F})_1$ such that $T_1\varpi T_2\cong T_1\omega T_2$.

Proof:

Let ω be defined by $\omega(p_2,s_2)=\varpi(\alpha(p_2,s_2)\in Q_2\times S(\tilde{F})_2$ where $\alpha: Q_2\times S(\tilde{F})_2\to S(\tilde{F})_2$ is a projection mapping by the definition and it is well-defined. Let ξ be an identity map on $S(\tilde{F})_2$ and η be an identity map on $Q_1\times Q_2$.

Then $\rho^{\varpi}(\eta(p_1,\mu^{t_i}(p_1),p_2,\mu^{t_i}(p_2)),s_2,(q_1,q_2))$

$$=\rho^{\varpi}((p_1,\mu^{t_i}(p_1),p_2,\mu^{t_i}(p_2)),s_2,(q_1,q_2))$$
$$=\rho_1(p_1,\mu^{t_i}(p_1),\varpi(s_2),q_1)\wedge(p_2,\mu^{t_i}(p_2),s_2,q_2))$$
$$=\rho_1(p_1,\mu^{t_i}(p_1),\omega(p_2,s_2),q_1)\wedge(p_2,\mu^{t_i}(p_2),s_2,q_2))$$
$$=\rho^{\omega}((p_1,\mu^{t_i}(p_1),p_2,\mu^{t_i}(p_2)),s_2,(q_1,q_2))$$
$$=\rho^{\omega}(\eta(p_1,\mu^{t_i}(p_1),p_2,\mu^{t_i}(p_2)),\xi(s_2),\eta(q_1,q_2))$$

Hence, $T_1\varpi T_2\cong T_1\omega T_2$.

Proposition 6.4:

Let $T_k=(Q_k,S(\tilde{F})_k,\rho_k)$ be GFSTS, $k=1,2$ and $\varpi: S(\tilde{F}_2)\to S(\tilde{F}_1)$ be a semigroup homomorphism. Then $T_1\varpi T_2$ is a GFSTS if and only if both T_1 and T_2 are GFSTS.

Proof:

Assume that T_1 and T_2 are GFSTS's. Since T_1 and T_2 are commutative and ϖ is homomorphism, therefore for all $(p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$ and $s_2, t_2 \in S_2$ where $i \geq 1$.

$$\rho^{\varpi}((p_1,\mu^{t_{i-1}}(p_1),p_2,\mu^{t_{i-1}}(p_2)),s_2t_2,(q_1,q_2))$$

$$= \rho_1(p_1,\mu^{t_{i-1}}(p_1),\varpi(s_2t_2),q_1) \wedge \rho_2(p_2,\mu^{t_{i-1}}(p_2),s_2t_2,q_2)$$

$$= \rho_1(p_1,\mu^{t_{i-1}}(p_1),\varpi(s_2)\varpi(t_2),q_1) \wedge \rho_2(q_2,\mu^{t_{i-1}}(q_2),s_2t_2,p_2)$$

$$= \rho_1(p_1,\mu^{t_{i-1}}(p_1),\varpi(t_2)\varpi(s_2),q_1) \wedge \rho_2(q_2,\mu^{t_{i-1}}(q_2),t_2s_2,p_2)$$

$$= \rho_1(p_1,\mu^{t_{i-1}}(p_1),\varpi(t_2s_2),q_1) \wedge \rho_2(p_2,\mu^{t_{i-1}}(p_2),t_2s_2,q_2)$$

$$= \rho^{\varpi}((p_1,\mu^{t_{i-1}}(p_1),p_2,\mu^{t_{i-1}}(p_2)),t_2s_2,(q_1,q_2))$$

Thus $T_1 \omega T_2$ is commutative. Let $(p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$ and $s_2 \in S(\tilde{F})_2$ where $i \geq 0$.

$$\rho^{\varpi}((p_1,\mu^{t_i}(p_1),p_2,\mu^{t_i}(p_2)),s_2,(q_1,q_2))$$
$$= \rho_1(p_1,\mu^{t_i}(p_1),\varpi(s_2),q_1) \wedge \rho_2(p_2,\mu^{t_i}(p_2),s_2,q_2)$$

Then as T_1 and T_2 are switching, hence

$$\rho_1(p_1,\mu^{t_i}(p_1),\varpi(s_2),q_1) = \rho_1(q_1,\mu^{t_i}(q_2),\varpi(s_2),p_1)$$

$$\rho_2(p_2,\mu^{t_i}(p_2),s_2,q_2) = \rho_2(q_2,\mu^{t_i}(q_2),s_2,p_2)$$

thus,

$$\rho^{\varpi}((p_1,\mu^{t_i}(p_1),p_2,\mu^{t_i}(p_2)),s_2,(q_1,q_2))$$

$$= \rho_1(p_1,\mu^{t_i}(p_1),\varpi(s_2),q_1) \wedge \rho_2(p_2,\mu^{t_i}(p_2),s_2,q_2)$$

$$= \rho_1(q_1,\mu^{t_i}(q_2),\varpi(s_2),p_1) \wedge \rho_2(q_2,\mu^{t_i}(q_2),s_2,p_2)$$

$$= \rho^{\varpi}((q_1,\mu^{t_i}(q_1),q_2,\mu^{t_i}(q_2)),s_2,(p_1,p_2))$$

Therefore, $T_1 \omega T_2$ is switching. Conversely, assume that $T_1 \omega T_2$ is a GFSTS.

Let $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $s_2, t_2 \in S(\tilde{F})_2$ where $i \geq 1$. Since $T_1 \omega T_2$ is commutative.

$$\rho_1(p_1, \mu^{t_{i-1}}(p_1), \varpi(s_2)\varpi(t_2), q_1) \wedge \rho_2(q_2, \mu^{t_{i-1}}(q_2), s_2t_2, p_2)$$

$$= \rho_1(p_1, \mu^{t_{i-1}}(p_1), \varpi(s_2t_2), q_1) \wedge \rho_2(p_2, \mu^{t_{i-1}}(p_2), s_2t_2, q_2)$$

$$= \rho^{\varpi}((p_1, \mu^{t_{i-1}}(p_1), p_2, \mu^{t_{i-1}}(p_2)), s_2t_2\ (q_1, q_2))$$

$$= \rho^{\varpi}((p_1, \mu^{t_{i-1}}(p_1), p_2, \mu^{t_{i-1}}(p_2)), t_2s_2, (q_1, q_2))$$

$$= \rho_1(p_1, \mu^{t_{i-1}}(p_1), \varpi(t_2s_2), q_1) \wedge \rho_2(p_2, \mu^{t_{i-1}}(p_2), t_2s_2, q_2)$$

Then, it shows that $\rho_1(p_1, \mu^{t_{i-1}}(p_1), \varpi(s_2t_2), q_1) = \rho_1(p_1, \mu^{t_{i-1}}(p_1), \varpi(t_2s_2), q_1)$ and $\rho_2(p_2, \mu^{t_{i-1}}(p_2), s_2t_2, q_2) = \rho_2(p_2, \mu^{t_{i-1}}(p_2), t_2s_2, q_2)$ whereby, T_2 is commutative.

Example 6.3.3.1: (Restricted cascade product of GFSTS)

Let $T_1 = (Q_1, S(\tilde{F})_1, \rho_1)$ and $T_2 = (Q_2, S(\tilde{F})_2, \rho_2))$ be GFSTS's, where $Q_1 = \{p_1, p_2\}, Q_2 = \{q_1, q_2\}, S(\tilde{F})_1 = \{s_1, t_1, u\}$, $, S(\tilde{F})_2 = \{s_2, t_2\}$ and ρ_1 and ρ_2 are defined as follows:

$$\rho_1(p_1, \mu^{ti}(p_1), s_1, p_1) = 0.5$$
$$\rho_1(p_2, \mu^{ti}(p_2), s_1, p_1) = 0.2$$
$$\rho_1(p_1, \mu^{ti}(p_1), t_1, p_2) = 0.2$$
$$\rho_1(p_2, \mu^{ti}(p_2), t_1, p_2) = 0.6$$
$$\rho_1(p_1, \mu^{ti}(p_1), u_1, p_2) = 0.4$$
$$\rho_2(p_2, \mu^{ti}(p_2), u_1, p_1) = 0.7$$
$$\rho_2(q_1, \mu^{ti}(q_1), s_2, q_1) = 0.6$$
$$\rho_2(q_2, \mu^{ti}(q_2), s_2, p_1) = 0.2$$
$$\rho_2(q_1, \mu^{ti}(q_1), t_2, q_2) = 0.5$$
$$\rho_2(q_2, \mu^{ti}(q_2), t_2, q_1) = 0.35$$

Now the function $\varpi: S(\tilde{F})_1 \to S(\tilde{F})_2$ is defined as

$\varpi(s_2) = s, \varpi(t_2) = t.$

Next, define the partial function $\rho^{\varpi}: (Q_1 \times Q_2) \times S(\tilde{F})_2 \times (Q_1 \times Q_2) \to [0,1]$ as:

$$\rho^{\varpi}(p_1, \mu^{ti}(p_1), q_1, \mu^{ti}, (q_1)), s_2, (p_1, q_1))$$
$$= \rho_1(p_1, \mu^{ti}(p_1), \varpi(s_2), p_1) \wedge \rho_2(q_1, \mu^{ti}(q_1), s_2, q_1) = 0.5$$
$$\rho^{\varpi}(p_1, \mu^{ti}(p_1), q_2, \mu^{ti}, (q_2)), s_2, (p_1, q_1))$$
$$= \rho_1(p_1, \mu^{ti}(p_1), \varpi(s_2), p_1) \wedge \rho_2(q_2, \mu^{ti}(q_2), s_2, q_1) = 0.2$$
$$\rho^{\varpi}(p_2, \mu^{ti}(p_2), q_1, \mu^{ti}, (q_1)), s_2, (p_1, q_1))$$
$$= \rho_1(p_2, \mu^{ti}(p_2), \varpi(s_2), p_1) \wedge \rho_2(q_1, \mu^{ti}(q_1), s_2, q_1) = 0.2$$
$$\rho^{\varpi}(p_2, \mu^{ti}(p_2), q_2, \mu^{ti}, (q_2)), s_2, (p_1, q_1))$$
$$= \rho_1(p_2, \mu^{ti}(p_2), \varpi(s_2), p_1) \wedge \rho_2(q_2, \mu^{ti}(q_2), s_2, q_1) = 0.2$$
$$\rho^{\varpi}(p_1, \mu^{ti}(p_1), q_1, \mu^{ti}, (q_1)), t_2, (p_2, q_2))$$
$$= \rho_1(p_1, \mu^{ti}(p_1), \varpi(t_2), p_2) \wedge \rho_2(q_1, \mu^{ti}(q_1), t_2, q_2) = 0.2$$
$$\rho^{\varpi}(p_1, \mu^{ti}(p_1), q_2, \mu^{ti}, (q_2)), t_2, (p_2, q_1))$$
$$= \rho_1(p_1, \mu^{ti}(p_1), \varpi(t_2), p_2) \wedge \rho_2(q_2, \mu^{ti}(q_2), t_2, q_1) = 0.2$$
$$\rho^{\varpi}(p_2, \mu^{ti}(p_2), q_1, \mu^{ti}, (q_1)), t_2, (p_2, q_2))$$
$$= \rho_1(p_2, \mu^{ti}(p_2), \varpi(t_2), p_2) \wedge \rho_2(q_1, \mu^{ti}(q_1), t_2, q_2) = 0.5$$
$$\rho^{\varpi}(p_2, \mu^{ti}(p_2), q_2, \mu^{ti}, (q_2)), t_2, (p_2, q_1))$$
$$= \rho_1(p_2, \mu^{ti}(p_2), \varpi(t_2), p_2) \wedge \rho_2(q_2, \mu^{ti}(q_2), t_2, q_1) = 0.35$$

And δ^{ϖ} is 0 elsewhere. It follows that $M_1 \varpi M_2 \cong M_1 \omega M_2$ is restricted cascade product.

6.4 General Fuzzy Switchboard Polytransformation Semigroup

Transformation semigroup is utmost importance for semigroup theory, where every semigroup is isomorphic to a transformation semigroup (Linton et al., 2002). Since there is more than one transformation semigroup in the system, thus it is called as poly-transformation semigroup. The definitions of poly-transformation semigroup in GFSA are introduced. Some properties are examined and the provings are shown.

Definition 6.8:

A polytransformation semigroup (pts) is a triple $T = (Q, S(\tilde{F}), \gamma)$ where

Q is a finite nonempty set, $S(\tilde{F})$ is a finite semigroup of $\tilde{F}$, γ is a fuzzy subset of $(Q \times [0,1]) \times S(\tilde{F}) \to P(Q \times [0,1]) \backslash \{\emptyset\}$.

Such that,

i. $\gamma(\gamma(p, \mu^{t_i}(p), u), v) = \gamma(p, \mu^{t_i}(p), uv) \forall p \in Q, u, v \in S(\tilde{F})$ and $\gamma(P, u) = \cup \{\gamma(p, \mu^{t_i}(p), u) | p \in P\}$, $P \subseteq Q$ and $i \geq 0$.

ii. If $S(\tilde{F})$ contains the identity e, then $\gamma(p, \mu^{t_i}(p), e) = \{p\}\ \forall p \in Q$ and $i \geq 0$.

If the property holds, then $T = (Q, S(\tilde{F}), \gamma)$ is called **faithful**.

iii. Let $u, v \in S(\tilde{F})$, if $\gamma(p, \mu^{t_i}(p), u) = \gamma(p, \mu^{t_i}(p), v), \forall p \in Q$ then $u = v$.

Definition 6.9:

An anti-polytransformation semigroup (pts) is a triple $T = (Q, S(\tilde{F}), \gamma)$ where

Q is a finite nonempty set, $S(\tilde{F})$ is a finite semigroup of $\tilde{F}$, γ is a fuzzy subset of $(Q \times [0,1]) \times S(\tilde{F}) \to P(Q \times [0,1]) \backslash \{\emptyset\}$.

Such that,

i. $\gamma(\gamma(p, \mu^{t_i}(p), u), v) = \gamma(p, \mu^{t_i}(p), vu) \forall p \in Q, u, v \in S(\tilde{F})$ and $\gamma(P, u) = \cup \{\gamma(p, \mu^{t_i}(p), u) | p \in P\}$, $P \subseteq Q$ and $i \geq 0$.

ii. If $S(\tilde{F})$ contains the identity e, then $\gamma(p, \mu^{t_i}(p), e) = \{p\}\ \forall p \in Q$ and $i \geq 0$.

If the property holds, then $T = (Q, S(\tilde{F}), \gamma)$ is called **faithful**.

iii. Let $u, v \in S(\tilde{F})$, if $\gamma(p, \mu^{t_i}(p), u) = \gamma(p, \mu^{t_i}(p), v), \forall p \in Q$ then $u = v$.

To summarize, if the definition of polytransformation semigroup is equal to the anti-polytransformation semigroup, then it is commutative properties.

Theorem 6.0:

Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, then there exists a faithful anti-polytransformation semigroup with identity, denote by $T = (Q, S(\tilde{F}), \gamma)$.

Proof:

Define $\gamma: (Q \times [0,1]) \times S(\tilde{F}) \to P(Q \times [0,1]) \backslash \{\emptyset\}$ by

$$\gamma(p, \mu^{t_i}(p), S_x) = \{q \in Q | S_x(p, \mu^{t_i}(p)) = q, \mu^{t_i}(q)\}$$

$$= \{q, \mu^{t_i}(q)\} \subseteq P(Q \times [0,1]) \backslash \{\emptyset\}$$

In order to prove γ is well defined, let $i \geq 0, S_x, S_y \in S(\tilde{F})$ and $S_x = S_y$. Therefore, $S_x\big(p, \mu^{t_i}(p)\big) = S_y\big(p, \mu^{t_i}(p)\big) \forall p \in Q \leftrightarrow \gamma(p, \mu^{t_i}(p), S_x) = \gamma(p, \mu^{t_i}(p), S_y)$. Hence, γ is well defined.

Let $S_x, S_y \in S(\tilde{F}), p \in Q$ and $i \geq 0$.

Next, $\gamma\big(\gamma(p, \mu^{t_i}(p), S_x), S_y\big) = \gamma(\{q, \mu^{t_i}(q)\}, S_y)$, where q is such that $S_x(p) = q, q \in Q, i \geq 0$.

$$= \cup \{\gamma(q, \mu^{t_i}(q), S_y) | q \in \{q\}$$

$$= \{S_y(q, \mu^{t_i}(q))\}$$

$= \{r, \mu^{t_i}(r)\}$, where r is such that $S_y(q) = r, r \in Q$

Also,

$$\gamma\big(\gamma\big(p, \mu^{t_i}(p)\big), S_y \circ S_x\big) = \{\big(S_y \circ S_x\big)\big(p, \mu^{t_i}(p)\big)\}$$

$$= \{S_y(S_x\big(p, \mu^{t_i}(p)\big))\}$$

$= \{S_y(q, \mu^{t_i}(q))\}$, since $S_x(p) = q, q \in Q$

$= \{r, \mu^{t_i}(r)\}$, where r is such that $S_y(q) = r, r \in Q$

From the definition 6.9(i) in anti-polytransformation semigroup, thus $\gamma\big(\gamma(p, \mu^{t_i}(p), S_x), S_y\big) = \gamma(p, \mu^{t_i}(p), S_y \circ S_x)$.

S_λ is the identity element in $S(\tilde{F})$. Next, $\gamma(p, \mu^{t_i}(p), S_\lambda) = \big\{S_\lambda\big(p, \mu^{t_i}(p)\big)\big\} = \{p, \mu^{t_i}(p)\} \forall p \in Q$.

Hence, this is the definition 6.9(ii).

Let $S_x, S_y \in S(\tilde{F}), p \in Q$ and $\gamma(p, \mu^{t_i}(p), S_x) = \gamma(p, \mu^{t_i}(p), S_y)$. Therefore, $\{S_x(p, \mu^{t_i}(p))\} = \{S_y(p, \mu^{t_i}(p))\} \leftrightarrow S_x(p) = S_y(p)$. Since p is arbitrary, $S_x = S_y$.

Hence, this is definition 6.9(iii).

Thus $(Q, S(\tilde{F}), \gamma)$ is an anti-polytransformation semigroup.

Definition 6.10:

Let $T = (Q, S(\tilde{F}), \gamma)$ be a polytransformation semigroup (pts). Then,

i. T is commutative if it satisfied $\gamma(p, \mu^{t_{i-1}}(p), uv) = \gamma(p, \mu^{t_{i-1}}(p), vu)$, $\forall p \in Q, \forall u, v \in S(\tilde{F})\,, i \geq 1$.

ii. T is switching if it satisfied $\rho(q, \mu^{t_i}(q), uv, p) = \rho(p, \mu^{t_i}(p), uv, q)$, $\forall q, p \in Q, \forall u, v \in S(\tilde{F})\,, i \geq 0$.

If T satisfied both conditions which are commutative and switching, thus it is called as **General Finite Switchboard Polytransformation Semigroup (GFSPS).**

Lemma 6.3:

$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ is a commutative GSPS for every $i \geq 1$, $\gamma(q, \mu^{t_{i-1}}(q), uv) = \gamma(q, \mu^{t_{i-1}}(q), vu)$, for all $q \in Q$, $u \in S(\tilde{F})\,, v \in S(\tilde{F})^*$.

Proof:

If $\tilde{F}^*$ is commutative, clearly T is also commutative. Denote $\rho(q, \mu^{t_{i-1}}(q), uv) = \rho(q, \mu^{t_{i-1}}(q), vu)$, where $\forall q, p \in Q$, $u, v \in S(\tilde{F})\,, i \geq 1$.

Let $v \in S(\tilde{F})^*$ and $|v| = n$. If $n = 0$, then $v = \lambda$. Hence,

$$\rho(q, \mu^{t_{i-1}}(q), uv) = \rho(q, \mu^{t_{i-1}}(q), u\lambda)$$

$$= \rho(q, \mu^{t_{i-1}}(q), u)$$

$$= \rho(q, \mu^{t_{i-1}}(q), \lambda u)$$

$$= \rho(q, \mu^{t_{i-1}}(q), vu)$$

Now suppose the result is true for all $x \in S(\tilde{F})^*$ such that $|x| = n - 1, n > 0$. Let $u = xy$ where $y \in S(\tilde{F})$ and $x \in S(\tilde{F})^*$. Then,

$$\rho^*(q, \mu^{t_{i-1}}(q), uv) = \rho^*(q, \mu^{t_{i-1}}(q), xyv)$$

$$= \rho^*(q, \mu^{t_{i-1}}(q), xvy)$$

$$= \rho^*(q, \mu^{t_{i-1}}(q), vxy)$$

$$= \rho^*(q, \mu^{t_{i-1}}(q), vu)$$

The lemma 6.3 completes the proof and the result now follows by induction.

Lemma 6.4:

$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ is a switching GSPS for every $i \geq 0$,

$\rho(q, \mu^{t_i}(q), uv, p) = \rho(p, \mu^{t_i}(p), uv, q)$, for all $q, p \in Q$, $u \in S(\tilde{F})$, $v \in S(\tilde{F})$ *.

Proof:

Let $u, v \in S(\tilde{F})$ * and $|uv| = n$. If $n = 0$, then $uv = \lambda$.

$$\rho(q, \mu^{t_i}(q), uv, p) = \bigvee_{r \in Q} \{\rho(q, \mu^{t_i}(q), u, r) \wedge \rho(r, \mu^{t_{i+1}}(r), v, p)\}$$

$$= \bigvee_{r \in Q} \{\rho(r, \mu^{t_i}(r), u, q) \wedge \rho(p, \mu^{t_{i+1}}(p), v, r)\}$$

$$= \bigvee_{r \in Q} \{(p, \mu^{t_i}(p), v, r) \wedge \rho(r, \mu^{t_{i+1}}(r), u, q\}$$

$$= \rho^*(p, \mu^{t_i}(p), vu, q)$$

$$= \rho^*(p, \mu^{t_i}(p), vu, q)$$

Hence, the result is true and follows by induction.

6.5 Summary

The concept of transformation semigroup, covering, cascade product and direct product plays an important role in the study of automata. There are numerous classes of the semigroup. Semigroup can be considered as a group if it consists of monoid and inverse element. Since transformation semigroup has a huge number of sets of states, it is easier to explore the space of all possible finite state machines by listing these semigroups. The concept of covering is more useful in transformation semigroup of general fuzzy switchboard automata. For instance, GFSTS of A state machine can be covered by GFSTS of B state machine if the properties are satisfied. In this chapter, the direct product and cascade product in General Fuzzy Switchboard Transformation Semigroup (GFSTS) are examined. These products are the basic general mathematical construction and usefulness in the algebraic automata theory.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Introduction

In this chapter, the results are concluded and discussed. Some limitations and recommendations about this research are provided. Meanwhile, the theory and the properties used are applied to different platforms.

7.2 Discussion and findings

The algebraic properties and topological properties are important in the operations and the whole system. In order to make the system or machine functions well, the properties must be satisfied and understandable. The theory is applied in the applications of real life in Finite Switchboard Automata in order to make it more understandable and interesting.

Nowadays, switching and commutative processing are central to the computation. However, some issues appeared when classical versions are unable to reflect the real needs of the current computer science, whereby they are unable to predict the flow of the next input information into a designated output. Switchboard in a finite state machine acts as a controller to control the direct flow of information from one state to another state and also plays an important role in communication between the subsystems. If the simple system fulfilled the two properties of the commutative state machine and switching state machine, thus the system is a finite switchboard state machine. Many researchers studied on the theory of Finite Switchboard Automata (FSA). However, the algebraic approach is still lacking. Thus, it is necessary to understand the modeling of switching mechanisms as a control device. The main purpose of this study is to investigate the algebraic properties in the finite switchboard automata. The properties of the switchboard automata which are commutative and switching are shown in Chapter 4. Two examples are provided which are Pac-man game and microwave and the properties of the switchboard automata are investigated. Based on these two examples, the Pac-man game is not a switchboard system because it does not satisfy the properties of the switchboard, meanwhile microwave is a switchboard system.

Fuzzy set theory is popular in solving control problems. By incorporating finite switchboard automata with fuzzy set theory, the algebraic properties are enhanced. This study aims to give a specific algorithm and also investigate the notion of Fuzzy Finite Switchboard Automata (FFSA) by the use of general algebraic structure, namely Complete Residuated Lattices (CRL). The theory of FFSA is extended to a more comprehensive structure by considering the membership values in a CRL. Many researchers studied CRL because it is a general algebraic structure with very important applications. CRL is applied in the concept of fuzzy set theory to obtain the membership values of the states. Fuzzy automata and language have gained more attention to the researchers in the wide-field application. By studying the fuzziness, such as fuzzy sets, fuzzy automata and language, the gap between the precision of formal language and the imprecision of natural language can be reduced. The algorithm and properties used are discussed in Chapter 4. An example of FFSA with CRL is provided in Chapter 4 to make a clear view. Therefore, Objective 1 which is to derive an efficient algebraic and the closure properties of FFSA is achieved.

Based on the idea from Doorstfatemeh and Kremer (2005) the notion of general fuzzy automata, the concept of General Fuzzy Switchboard Automata (GFSA) are introduced in Chapter 5. Topology studies the properties of spaces under any continuous deformation, such as stretching, bending and twisting without tearing or gluing. Sometimes, distances can be defined in these spaces which are called metric spaces. In order to make the states in the system connected and the machine operates functionally, one of the properties of topology, such as connectedness is studied. Furthermore, the newly defined Kuratowski fuzzy closure operator is used to establish fuzzy topology on a GFSA. The reason behind using GFA is because there is some possibility of overlapping transitions to the same state upon the same symbol from the different current states. Several different membership values at the same time known as multi-membership. GFA can also handle the application problems which were entirely dependent on fuzzy automaton as a modeling tool to assign membership values of active states of a fuzzy automaton, resolving the multi-membership, and defining and analyzing the continuous operation of the fuzzy automaton. The definition of GFSA is introduced and three examples are provided to differentiate the GFA and GFSA in order to make it in clear view and understanding. The properties and proof of GFSA are shown in Chapter 5. The concept of subsystem is usually applied in automata; however, it is first applied in GFSA. Thus, by considering the switchboard in GFA, the subsystem, strong subsystem and homomorphism of GFSA are introduced. New definitions and properties of GFSA are introduced. In order to make it more understandable, two applications of GFSA and their calculations are shown. Objective 2 of this research is achieved and all the properties and findings are discussed in Chapter 5.

A semigroup is an algebraic structure that shows a very close connection between self-adjoint operators. Transformation semigroups are important for the structure theory of finite state machines in automata theory. It defines all possible transitions set transformations that can be joined in time and also as a collection of the functions from a set to itself. Since they have a huge number of sets of states, it is easier to explore the space of all possible finite computations by listing these semigroups. General Fuzzy Switchboard Transformation Semigroup (GFSTS) is introduced by cooperating with the switchboard properties and the General Fuzzy Automata (GFA) in transformation semigroup. Some related definitions and properties are introduced and proved in Chapter 6. Since the algebraic product is an effective way to study in the state machine and automata theory, thus some algebraic products, such as covering, direct product and cascade products are combined in GFSTS and their properties are shown in Chapter 6. If the definition of polytransformation semigroup is equal to the anti-polytransformation semigroup, then it is commutative properties. Then, if General Fuzzy Polytransformation satisfied commutative and switching conditions, thus it is called General Fuzzy Switchboard Polytransformation Semigroup. Therefore, Objective 3 in this research is achieved.

7.3. Limitation of research

Based on the literature, it is difficult to find additional information regarding the topic. By referring to the examples of GFSA, the maximum membership values of certain states that are provided consist of two input symbols and two membership values only. However, it can be up to more than two input symbols and membership values.

7.4 Suggestions of future research

The study has shown that the switchboard automata is widely known in the automata field. According to the literature, many researchers studied switchboard state machines or also known as switchboard automata on many different platforms. Based on the current research and conclusions, some recommendations for future research are suggested. First and foremost, Neutrosophic General Finite Automata is studied as preliminary work. The future research will be more interesting by considering switchboard state machine in Neutrosophic General Finite Automata and known as Neutrosophic General Finite Switchboard Automata. Besides that, another method, such as Lukasiewicz algebras and Heyting algebras can be used to obtain the membership value of states instead of CRL. In order to improve the operations of the system, future study is suggested to extend the topological properties in automata.

7.5 Ending remarks

The previous chapter already mentioned that the switchboard automata is an important mechanism in computer science. It is able to make the system operate functionally. However, the properties between the subsystems must be satisfied and understandable in order to make them function well. The objectives of this research are achieved and the entire algorithms are discussed in Chapter 3.

REFERENCES

Abdullah, S., Naz, R. and Pedrycz, W. (2017). Cubic finite state machine and cubic transformation semigroups. *New Trends in Mathematical Sciences, 4*, pp. 24 – 39.

Abolpour, K. and Zahedi, M. M. (2012). Isomorphism between two BL- General fuzzy automata. *Soft Computing, 16*, pp. 729 – 736.

Abolpour, K. and Zahedi, M. M. (2017). General fuzzy automata based on complete residuated lattice-valued. *Journal of Fuzzy Systems*, *14*, pp. 103-121.

Alur, R. and Dill, D. L. (1994). A theory of timed automata. *Theoretical Computer Sciences*, *126(2)*, pp. 183 – 235.

Ather, D. Singh, R. and Katiyar, V. (2013). An algorithm to design finite automata that accept strings over input symbol a and b having exactly x number of a & y number of b. *2013 International Conference on Information Systems and Computer Networks.* Mathura, India. Institute of Electrical and Electronics Engineers. pp. 1 – 4.

Belohlavek, R. (2002). *Fuzzy Relational Systems: Foundations and Principles*. New York: Springer US.

Belohlavek, R. (2003). Some properties of residuated lattices. *Czechoslovak Mathematical Journal*, *53(1)*, pp. 161 – 171.

Blount, K. and Tsinakis, C. (2003). The structure of residuated lattices. *International Journal of Algebra and Computation, 13(4)*, pp. 437 – 461.

Broy, M. and Wirsing, M. (2000). Algebraic state machines. *International Conference on Algebraic Methodology and Software Technology*. Berlin, Heidelberg. Springer. pp. 89 – 118.

Busneag, D. and Piciu, D. (2015). A new approach for classification of filters in residuated lattices. *Fuzzy Sets and Systems, 260*,pp. 121 – 130.

Ciric, M., Ignjatovic, J. and Damljanovi'c, N. (2012). Bisimulations for fuzzy automata. *Fuzzy Sets and Systems, 186*, pp. 100 – 139.

Derus, K. Md., Kavikumar, J. and Amir Hamzah, N. S. (2017). Decomposition of bipolar fuzzy finite state machine and transformation semigroups. *Journal of Engineering and Applied Sciences*, *12(3)*, pp. 679 – 683.

Dilworth, R. P. (1938). Abstract residuation over lattices. *Bulletin of the American Mathematical Society*, *44*, pp. 262 – 268.

Doner, J. (1970). Tree acceptors and some of their applications. *Journal of Computer and System Sciences*, *4(5)*, pp. 406 – 451.

Doostfatemeh, M. and Kremer, S. C. (2005). New directions in fuzzy automata, *International Journal of Approximate Reasoning*, *38*, pp. 175 – 214.

Doostfatemeh, M. and Kremer, S. C. (2006). General fuzzy automata, new efficient acceptors for fuzzy languages. *Institute of Electrical and Electronic Engineers International Conference on Fuzzy Systems*, pp. 2097 – 2103.

Gautam, V., Tiwari, S. P., Pal, P. and Tripathi, T. (2018). Categories of automata and languages based on a complete residuted lattice. *New Mathematics and Natural Computation*, *14(3)*, pp. 423 – 444.

Ghorani, M. and Zahedi, M. M. (2012). Characterizations of complete residuated lattice-valued finite tree automata. *Fuzzy Sets and Systems*, *199*, pp. 28 – 46.

Glabbeek, R. V. and Ploeger, B. (2008). Five determinisation algorithm. *International Conference on Algebraic Methodology and Software Technology*. Berlin, Heidelberg. Springer. pp. 161 – 170.

Gomez, M, Lizasoain, I. and Moreno, C. (2012). Lattice- valued finite state machines and lattice- valued transformation semigroups. *Fuzzy Sets and Systems*, *208*, pp. 1 – 21.

Gribkoff, E. (2013). Applications of deterministic finite automata. from https://web.cs.ucdavis.edu/~rogaway/classes/120/spring13/eric-dfa.pdf.

Guo, X. (2012). A comment on automata theory based on complete residuated lattice-valued logic: Pushdown automata. *Fuzzy Sets and Systems*, *199*, pp. 130 – 135.

Holcombe, W. M. L. (1982). *Algebraic Automata Theory*. United Kingdom: Cambridge University, Press Cambridge.

Horry, M. and Zahedi, M. M. (2013). Some (fuzzy) topologies on general fuzzy automata. *Iranian Journal of Fuzzy Systems*, *10*, pp. 73 – 89.

Horry, M. and Zahedi, M. M. (2009). Uniform and semi-uniform topology on general fuzzy automata. *Iranian Journal of Fuzzy Systems*, *6(2)*, pp. 19 – 29.

Horry, M. (2016a). Bipolar general fuzzy automata. *Journal of Linear and Topological Algebra. 5(2)*, pp. 83 – 91.

Horry, M. (2016b). Irreducibility on general fuzzy automata. *Iranian Journal of Fuzzy Systems. 13*, pp. 131 – 144.

Horry, M. (2017). Application of a group in general fuzzy automata. *Algebraic Structures and their Applications, 4(2)*, pp. 57 – 69.

Hopcorf, J. E. and Ullman, J. D. (1979). *Introduction to Automata Theory, Languages and Computation*, 1^{st} ed. United States: Addison Wesley.

Hopcorf, J. E., Motwani, R. and Ullman, J. D. (2000). *Introduction to Automata Theory, Languages and Computation*. 2^{nd} ed. United States: Addison Wesley.

Ignjatovic, J., Ciric, M. and Bogdanovic, S. (2008). Determinization of fuzzy automata with membership values in complete residuated lattices. *Information Sciences*, *178*, pp. 164 – 180.

Ignjatovic, J., Ciric, M., Bogdanovic, S. and Pethovic, T. (2010). Myhill-nerode type theory for fuzzy languages and automata. *Fuzzy Sets and System*, *161*, pp. 1288 – 1324.

Ignjatovic, J., Ciric, M. and Simovic, V. (2013). Fuzzy relation equations and subsystems of fuzzy transition systems. *Knowledge Based System*, *38*, pp. 48 – 61.

Inagaki, Y. (2002). On synchronized evolution of the network of automata. *IEEE Transcactions on Evolutionary Computation*, *6(2)*, pp. 147 – 158.

Ito, M. (2004). *Algebraic Theory of Automata and Languages*. Japan: World Scientific, Kyoto Sangyo University.

Jancic, Z. and Ciric, M. (2014). Brzozowski type determinization for fuzzy automata. *Fuzzy Sets and Systems*. Retrieved February 8, 2014, from doi: 10.1016/j.fss.2014.02.021.

Jancic, Z., Micic, I., Ignjatovic, J. and Ciric, M. (2016). Further improvements of determinization method for fuzzy finite automata, *Fuzzy Sets and Systems*, *301*, pp. 79 – 102.

Jasem, M. and Bratislava. (2007). On ideals of lattice ordered monoids. *Mathematica Bohemica*, *132(4)*, pp. 369 – 387.

Jin, J., Li, Q. and Li, Y. (2013). Algebraic properties of *L*-fuzzy finite automata. *Information Sciences- Informatics and Computer Science*, *Intelligent Systems*. *234*, pp. 182 – 202.

Jun, Y. B. (2005). Intuitionistic fuzzy finite state machines. *Journal Applied Mathematics computation*, *17(2)*, pp. 109 – 120.

Jun, Y. B. (2006). Intuitionistic fuzzy finite switchboard state machines. *Journal Applied Mathematics Computation*, *20(2)*, pp. 315 – 325.

Jun, Y. B., and Kavikumar, J. (2011). Bipolar fuzzy finite state machine. *Bulletin of the Malaysian Mathematical Sciences Society*, *34(1)*, pp. 181 – 188.

Kavikumar, J., Khamis, A. and Roslan, R. (2012), Bipolar-valued fuzzy finite switchboard state machines. *Lecture Notes in Engineering and Computer Science*, *2200*, pp. 571 – 576.

Kavikumar, J., Khamis, A. and Rusiman, M. S. (2013). *N*-structures applied to finite state machines. *International Journal of Applied Mathematics. 43(4)*, pp. 233 – 237.

Kavikumar, J., Tiwari, S. P., Amir Hamzah, N. S. and Sharan, Dr. S. (2019). Restricted cascade and wreath products of fuzzy finite switchboard state machines. *Iranian Journal of Fuzzy Systems*, *16(1)*, pp. 75 – 88.

Khan, Q., Mahmood, T., Ullah, K. and Jan, N. (2018). Single valued neutrosophic finite state machine and switchboard state machine. *New Trends in Neutrosophic Theory and Application*, *2*, Retrieved January 24, 2018, from http://doi.org/10.5281/zenodo.1237952

Khamirrudin, Md. D., Kavikumar, J. and Amir Hamzah, N. S. (2017). Decomposition of bipolar fuzzy finite state machines and transformation semigroup, *Journal of Engineering and Applied Sciences*, *12(3)*, pp. 679 – 683.

Konecny, J. and Krupka, M. (2017). Complete relations on fuzzy complete lattices. *Fuzzy Sets and Systems*, *320*, pp. 64 – 80.

Lawson, M. V. (2004). *Finite Automata*. London: Chapman and Hall

Li, Y. (2011). Finite Automata Theory with Membership values in lattices. *Information Sciences*, *181*, pp. 1003 – 1017.

Li, Y. M. and Pedrycz, W. (2005). Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoids. *Fuzzy Sets and Systems*, *156*, pp. 68 – 92.

Mahmood, T. and Khan, Q. (2016). Interval neutrosophic finite switchboard state machines, *Afrika Matematika, 27*, pp. 1361 – 1376.

Malik, D. S., Moderson, J. N. and Sen, M. K. (1994). On subsystems of fuzzy finite state machines, *Fuzzy Set and Systems, 68(1),* pp. 83 – 92.

Malik, D. S., Moderson, J. N. and Sen, M. K. (1997). Products of fuzzy finite state machines, *Fuzzy Sets and Systems,* 92*(1),* pp. 95 – 102.

McMullen, C. (2013). *Topology*. Cambridge: Harvard University.

Meduna, A. and Zemek, P. (2012). Jumping finite automata. *International Journal of Foundations of Computer Science. 23(7),* pp. 1555 – 1578.

Micic, I., Jancic, Z., Ignjatovic, J. and Ciric, M. (2015), Determinization of fuzzy automata by means of the degrees of language inclusion. *Institute of Electrical and Electronic Engineers Transactions on Fuzzy Systems, 23(6),* pp. 2144 – 2153.

Mordeson, J. N., and Malik, D. S. (2002). *Fuzzy Automata and Languages, Theory and Applications,* 1^{st} ed. London: Chapman and Hall.

Pan, H., Cao, Y., Zhang, M. and Chen, Y. (2014). Simulation for lattice-valued doubly labeled transitions systems. *International Journal of Approximate Reasoning, 55,* pp. 797 – 811.

Pan, H., Li, Y., Cao, Y. and Li, P. (2017). Nondeterministic fuzzy automata with membership values in complete residuated lattices, *International Journal of Approximate Reasoning, 82,* pp. 22 – 38.

Pedrycz, W. and Gacek, A. (2001). Learning of fuzzy automata. *International Jornal Computatation. Intelligent Application, 1(1),* pp. 19 – 33.

Qiu, D. W. (2001). Automata theory based on completed residuated lattice-valued logic(I). *Science in China Series: Information Sciences, 44,* pp. 419 – 429.

Qiu, D. W. (2002). Automata theory based on completed residuated lattice- valued logic(II). *Science in China Series*:*Information Sciences, 45*, pp. 442 – 452.

Qiu, D. W. (2006). Pumping lemma in automata theory based on completed residuated lattice- valued logic: a note. *Fuzzy Sets and Systems*, *15(1*), pp. 2128 – 2138.

Rabin, M., and Scott, D. (1959). Finite automata and their decision problems. *IBM Journal of Research and Development, 3,* pp. 114 – 125.

Reena, K. (2019). Bipolar vague finite switchboard state machine. *International Journal of Mathematics Trends and Technology, 65(7),* pp. 31 – 40.

Saeidi, R. A. and Shamsizadeh, M. (2019). Transformation of BL-general fuzzy automata. *International Journal Industrial Mathematics, 11(3),* pp. 177 – 187.

Santos, E. S. (1968). Maximin automata. *Information and Control*, *13*, pp. 363 – 377.

Santos, E. S. (1976). Fuzzy automata and language. *Information sciences*, *10(2),* pp. 193 – 197.

Sato, Y. and Kuroki, N. (2002). Fuzzy finite switchboard state machine. *The Journal of Fuzzy Mathematics, 10*, pp. 863 – 873.

Sato, Y. (2003). *Finite Switchboard State Machines and Fuzzy Finite Switchboard State Machines*, Japan: Hyogo University of Teacher Education.

Shamsizadeh, M., Zahedi, M. M. and Abolpour, K. (2016). Bisimulation for BL-general fuzzy automata. *Iranian Journal of Fuzzy Systems, 13(4),* pp. 35 – 50.

Shamsizadeh, M. and Zahedi, M. M. (2016a). Intuitionistic general fuzzy automata. *Soft Computing*, *20(9),* pp. 3505 – 3519.

Shamsizadeh, M. and Zahedi, M. M. (2016b). Minimal and statewise minimal intuitionistic general *L*-fuzzy automata. *Iranian Journal of Fuzzy System, 13(7),* pp. 131 – 152.

Shamsizadeh, M. and Zahedi, M. M. (2019). Bisimulation of type 2 for BL-general fuzzy automata. *Soft Computing*, *23*, pp. 9843 – 9852.

Sharma, A. K. (2004). *Introduction to Set Theory*. New Delhi: Discovery Publishing House.

Sharma, B. K., Tiwari, S. P. and Sharan, Dr. S. (2016). On algebraic study of fuzzy multiset finite automata. *Fuzzy Information and Engineering*, *8(3),* pp. 315 – 327.

Shum, K. P. (2017). A note on kuratowski’s theorem and its related topic. *Advances in Pure Mathematics, 7,* pp. 383 – 406.

Singh, A. K., Pandey, S. and Tiwari, S. P. (2017). On algebraic study of type-2 fuzzy finite state automata. *Journal Fuzzy Set Valued Analysis, 2*, pp. 86 – 95.

Singh, T. B. (2019). *Introduction to Topology*. 1^{st}ed. Singapore: Springer.

Sipser, M., (2005). *Introduction to the Theory of Computation*, 2^{nd} ed. Bostan: Cengage Learning.

Subramaniyan, S. and Rajasekar, M. (2012) Homomorphism in bipolar fuzzy finite state machines. *International Mathematical Forum*, *7(31*), pp. 1505 – 1516.

Stanovsky, D. (2007). Commutative idempotent residuated lattices. *Czechoslovak Mathematical Journal*, *57(1),* pp. 191 – 200.

Tabak, J. (2011). *Algebra Sets, Symbols, and the Language of Thought*. Revised Edition. New York: Infobase Learning

Thatcher, W. and Wright, J. B. (1968). Generalized finite automata with an application to a decision problem of second- order logic. *Mathematical System Theory*. *2*, pp. 57 – 82.

Wee, W. G. and Fu, K. S. (1969). A formulation of fuzzy automata and its application as a model of learning systems, *Institute of Electrical and Electronic Engineers Transactions Systems Man Cybernetics*, *5*, pp. 215 – 223.

Wee, W. G. (1967). *On generalization of adaptive algorithm and application of the fuzzy sets concept to pattern classification.* Purdue University.: Ph.D. Thesis.

Williams, O. K. (1973). *The Algebra of Finite State Machine*. University Centre: Master Thesis.

Wu, L. and Qiu, D. W. (2010). Automata theory based on complete residuated lattice-valued logic: reduction and minimization. *Fuzzy Sets and Systems*, *161*, pp. 1635 – 1656.

Wu, L., Qiu, D. W., and Xing, H. (2012). Automata theory based on complete residuated lattice-valued logic: turing machines. *Fuzzy Sets and Systems*, *208*, pp. 43 – 66.

Xing, H., Qiu, D. W., and Liu, F. (2009). Automata theory based on complete residuated lattice-valued logic: pushdown. *Fuzzy Sets and Systems*, *160*, pp. 1125 – 1140.

Xing, H. and Qiu, D. W. (2009). Automata theory based on complete residuated lattice-valued logic: a categorical approach. *Fuzzy Sets and Systems*, *160*, pp. 2416 – 2428.

Yaqoob, N. and Abughazalah, N. (2019). Finite switchboard state machine based on cubic sets. *Hindawi Complexity.* Retrieved on July, 2019, from https://doi.org/10.1155/2019.12548735

Zadeh, L. A. (1965). Fuzzy Set, *Information and Control, 8*, pp. 338 – 353.

VITA

The author was born on May 21, 1993, in Johor, Malaysia. She went to SMK Nitar for her secondary school. She pursued her degree at the Universiti Tun Hussein Onn Malaysia (UTHM) in Batu Pahat and graduated with the Bachelor Degree in Sciences (Applied Mathematics) with honors in 2016. During her stay at the university, the author has taken part in numerous extra-curricular activities. She is also one of the Vice Chancellor's Award winner for being the top scorer with Cumulative Grade Point Average (CGPA) of 3.82.