NON-LINEAR WAVES MODULATION
IN A STENOSED ELASTIC TUBE

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ABSTRACT

By using stretched coordinate of boundary-value problem, we have studied the non-linear wave modulation in a stenosed elastic tube filled with an incompressible fluid. We showed that the governing equation can be reduced to non-linear Schrödinger equation with variable coefficient.
CHAPTER 1

INTRODUCTION

1.1 General Introduction

Due to its applications in arterial mechanics, the propagation of pressure pulses in fluid-filled distensible tubes has been studied by several researchers such as Pedley (1980) and Fung (1981). As far as the biological applications are concerned, most of the works on wave propagation in compliant tubes have considered small amplitude waves ignoring the nonlinear effects and focused on the dispersive character of waves (see, Atabek and Lew (1966), Rachev (1980) and Demiray (1992). However, when the nonlinear terms arising from the constitutive equations and kinematical relations are introduced, one has to consider either finite amplitude, or small-but-finite amplitude waves, depending on the order of nonlinearity.

Rudinger (1970), Ling and Atabek (1972), Anliker et al (1968) and Tait and Moodie (1984) observed the propagation of finite amplitude waves in fluid-filled elastic or viscoelastic tube by using the characteristics method to study the formation of shock. On the other hand, the propagation of small-but-finite amplitude waves in distensible tubes has been investigated by Johnson (1972), Hashizume (1985, 1988), and Yomosa (1987) by employing various asymptotic methods.

Later, in a series of works of Demiray and Antar (1997-2002), they treated
the artery as an incompressible, prestressed, thin and isotropic elastic tube and
the blood as an incompressible inviscid, viscous or layered fluid. Then, by using
the reductive perturbation method in the long-wave approximation, they obtained
KdV, Burgers’ and KdV-Burgers’ type equations, respectively.

Tay and co-workers (2006, 2007, 2008) studied the non-linear waves prop-
agation in a prestressed thin elastic tube with a symmetrical stenosis filled with
inviscid, viscous and Newtonian fluid with variable viscosity, they showed that the
governing equations can be reduced to forced Korteweg-de Vries, forced perturbed
Korteweg-de Vries and forced Korteweg-de Vries-Burgers equations, respectively.

The modulation of small-but-finite amplitude pressure waves in a fluid-
filled distensible, linear elastic tube has been examined by Ravindran and Prasad
(1979). They obtained the non-linear Schrödinger (NLS) equation equation.

Antar and Demiray (1999) studied non-linear wave modulation in a pre-
stressed thin elastic tube filled with an inviscid fluid, they showed that governing
equations can be reduced to NLS equation.

By using approximate equations for viscous fluid, Demiray (1999) inves-
tigated non-linear wave modulation in a prestressed thin elastic tube filled with
by using full viscous equations for viscous fluid. Both cases, he showed that the
governing equations can be reduced to dissipative NLS equation.

The NLS equation is the simplest representative equation describing the
self-modulation of one-dimensional monochromatic plane waves in dispersive me-
dia. It has a balance between the nonlinearity and dispersion.

In all previous modulation works, they treated the arteries as circularly
cylindrical long tubes with a constant cross-section, however due to decomposition of fat or cholesterol in artery over time, the artery become narrower and may have variable radius along the axis of the tube as given in Tay and co-worker works on non-linear wave propagation in such a composite medium.

As far as we know, none of the literature works dealt with wave modulation in a stenosed elastic tube yet. Hence, in the present work, considering the artery as an incompressible, prestressed, thin-walled elastic tube with a symmetrical stenosis and the blood as an inviscid fluid, we will study the amplitude modulation of non-linear waves in such a composite medium by using the reductive perturbation method. We hope to obtain the NLS type of equation. We then sought the progressive wave solution to the non-linear evolution equation obtained.

1.2 Objectives

Objectives of this study are:

(i) Study the amplitude modulation of the non-linear waves in a stenosed elastic tube filled with an inviscid fluid by the stretched coordinates of the boundary-value problem.

(ii) Find the progressive wave solutions for the non-linear evolution equations obtained.

1.3 Scopes of the Study

The artery is treated as an incompressible, prestressed, thin and long circularly cylindrical, elastic tube with a symmetrical stenosis as shown in Figure
2.1 and the blood as an incompressible inviscid fluid.

1.4 Importance of the Study

Since the stenosis is associated to some health problems such as hypertension, kidney failure, stroke and heart problem, it is hoped that this research will contribute to the derivation of mathematical model for the non-linear waves modulation in a stenosed artery. Then, finding the progressive wave solution. This result will be beneficial in the study of non-linear evolution equations and creating more experts in UTHM in particular and in Malaysia in general that require more expertise in the field of non-linear waves that will give more insight to the medical field.

1.5 Methodology

To derive the equations of tube and fluids, first of all the knowledge of tensor analysis and continuum mechanics in curvilinear coordinates is needed. Then to solve the field quantities in the governing equations (equations of tube and fluids), the knowledge of perturbation method is needed. By using the stretched coordinates and asymptotic expansion, the governing equations will be reduced to the non-linear evolution equations. Later, by using the progressive wave solution, the non-linear evolution equations obtained will be solved. Using matlab version 7, the graphical outputs of the progressive wave solutions will be plotted.
1.6 Outline of Thesis

The outline of thesis are as follows: Chapter 1 discusses the introduction of the research. Chapter 2 gives derivation of a stenosed elastic tube. Chapter 3 shows the equations of inviscid fluid. Chapter 4 studies wave modulation of a stenosed elastic tube filled with an inviscid fluid. Chapter 5 deals with the progressive wave solutions. Chapter 6 present the graphical output of Chapter 5.
CHAPTER 2

THE EQUATION OF A STENOSED ELASTIC TUBE

2.1 Introduction

The arterial wall material is known to be incompressible, viscoelastic and anisotropic (Gow and Taylor, 1968). However, for simplicity in mathematical analysis, it is assumed that the artery is incompressible, homogeneous, elastic and isotropic.

For a healthy human being, the systolic pressure is about 120 mm Hg and the diastolic pressure is around 80 mm Hg. This means that the arteries are subjected to a mean static pressure $P_0 = 100$ mm Hg and in the course of blood flow, a dynamical pressure increment $\Delta P = \pm 20$ mm Hg is added on this initial field. Moreover, experimental studies (Fung, 1984) revealed that the arteries are also subjected to an initial axial stretch $\lambda_2$, which is about $\lambda_2 = 1.6$. These observations show that the arteries are initially subjected to a static deformation both in the radial and the axial directions and a dynamical pressure (or a radial displacement $u^*(z^*, t^*)$) is superimposed on this initial deformation. Due to the external tethering (tissues holding the artery in its position) in the axial direction, the arteries do not have axial displacement. Or the effect of axial displacement can be neglected. Thus, under the light of the above description, the governing equation of a prestressed incompressible elastic and isotropic elastic tube which represents artery will be derived in this chapter.
2.2 Derivation of Equation of a Stenosed Elastic Tube

We consider a thin, symmetrical and long tube of circular cross-section with radius $R_0$ in the cylindrical polar coordinates $(R^*, \Theta, Z^*)$. Then, the position vector $\mathbf{R}$ of a point on the tube can be described by

$$\mathbf{R} = R_0 \mathbf{e}_r + Z^* \mathbf{e}_z,$$

where $\mathbf{e}_r$, $\mathbf{e}_\Theta$ and $\mathbf{e}_z$ are the unit base vectors in the cylindrical polar coordinates and $Z^*$ is the axial coordinates of a material point in the natural state.

The arclengths along the meridional and circumferential curves are given by

$$dS_Z = dZ^*, \quad dS_\Theta = R_0 d\Theta.$$  \hfill (2.2)

Motivated by the experimental observations (Fung, 1984), the elastic tube is assumed to be subjected to an axial stretch ratio $\lambda_2$ and the static pressure $P^*_0(Z^*)$. Thus, the deformation can be described by

$$r_0 = [r_0 - f^*(z^*)] \mathbf{e}_r + z^* \mathbf{e}_z, \quad z^* = \lambda_2 Z^*,$$

where $\lambda_2$ is the constant stretch ratio along the tube axis, $z^*$ is the axial coordinate after static deformation and $f(z^*)$ is the stenosis function after this static deformation. Hence, the arclengths after static deformation along the meridional and circumferential directions are given by

$$ds^0_Z = [1 + (-f''(z^*))^2]^{1/2} dz^*, \quad ds^0_\Theta = [r_0 - f^*(z^*)]d\Theta,$$

where the prime denotes the differentiation of the corresponding field variable with respect to $z^*$.

Next, the stretch ratios along the meridional and circumferential curves are given by

$$\lambda_1^0 = \frac{ds^0_Z}{dS_Z} = \lambda_2 [1 + (-f''(z^*))^2]^{1/2}, \quad \lambda_2^0 = \frac{ds^0_\Theta}{dS_\Theta} = \frac{[r_0 - f^*(z^*)]}{R_0}. $$  \hfill (2.5)
Upon this initial static deformation, a finite dynamical radial displacement \( u^*(z^*, t^*) \) is superimposed, where \( t^* \) is the time parameter, but, in view of the external tethering (as explain Section 4.1) in the axial direction, the axial displacement is assumed to be negligible. Then, the position vector \( \mathbf{r} \) of a generic point on the tube can be described by

\[
\mathbf{r} = [r_0 - f^*(z^*) + u^*(z^*, t^*)]\mathbf{e}_r + z^*\mathbf{e}_z.
\]  

(2.6)

![Diagram](image)

Figure 2.1: The geometry of the stenosed tube in various configuration

The arc lengths along the deformed meridional and circumferential curves are, respectively, given by

\[
\begin{align*}
    ds_z &= \left[1 + \left(-f^{**} + \frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2} \, dz^*, \\
    ds_\theta &= [r_0 - f^*(z^*) + u^*(z^*, t^*)]d\theta.
\end{align*}
\]  

(2.7)

Then, the stretch ratios along the meridional and circumferential curves in the final configuration read, respectively,

\[
\begin{align*}
    \lambda_1 &= \lambda_z[1 + \{-f^* + \frac{\partial u^*}{\partial z^*}\}^2]^{1/2}, \\
    \lambda_2 &= \frac{r_0 - f^*(z^*) + u^*(z^*, t^*)}{R_0}.
\end{align*}
\]

(2.8)
The unit tangent vector $t$ along the deformed meridional curve and the unit exterior normal vector $n$ to the deformed membrane are given by

$$
t = \frac{-f'' + \frac{\partial u^*}{\partial z^*}}{\Lambda} e_r + e_z, \quad n = \frac{e_r - \left[-f'' + \frac{\partial u^*}{\partial z^*}\right] e_z}{\Lambda},$$

(2.9)

where the function $\Lambda$ is defined by

$$\Lambda = \left[1 + \left(-f'' + \frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2}.$$

(2.10)

The material of the tube is assumed to be incompressible. This condition imposes the following restriction on the thickness $H$ and $h$, before and after final deformation respectively

$$h = \frac{H}{\lambda_1 \lambda_2}.$$

(2.11)

Let $T_1$ and $T_2$ be the membrane forces acting per unit length along the meridional and circumferential curves, respectively.

![Diagram of forces acting on a small tube element](image)

**Figure 2.2: Forces acting on a small tube element**

The total force acting on a tube element placed between the planes $z^* = \text{const}$, $z^* + dz^* = \text{const}$, $\theta = \text{const}$ and $\theta + d\theta = \text{const}$ is given by

$$d\mathbf{F} = -T_1 ds_{\theta}|z^* + T_1 ds_{\theta}|z^* + dz^* - T_2 ds_z|\theta + T_2 ds_z|\theta + dz^* + P ds_z ds_{\theta},$$

(2.12)
where $\mathbf{P}$ is the fluid reaction force acting per unit deformed area. The force vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ are defined by

$$
\mathbf{T}_1 = T_1 \mathbf{t}, \quad \mathbf{T}_2 = T_2 \mathbf{e}_\theta.
$$

(2.13)

Here $T_1$ and $T_2$ are the scalar forces resultant of the tube and they are assumed to be tangent to the membrane.

Summation of the first and second terms in equation (4.32) yields

$$
-T_1 ds_1 \big|_{z^*} + T_1 ds_2 \big|_{z^* + dz^*}
= -\frac{T_1}{\Lambda} \left[ \left( -f^* + \frac{\partial u^*}{\partial z^*} \mathbf{e}_r \right) + \mathbf{e}_z \right] [r_0 - f^*(z^*) + u^*] d\theta \big|_{z^*} \\
+ \frac{T_1}{\Lambda} \left[ \left( -f^* + \frac{\partial u^*}{\partial z^*} \mathbf{e}_r \right) + \mathbf{e}_z \right] [r_0 - f^*(z^*) + u^*] d\theta \big|_{z^* + dz^*} \\
= \frac{\partial}{\partial z^*} \left\{ \frac{T_1}{\Lambda} \left[ \left( -f^* + \frac{\partial u^*}{\partial z^*} \right) \mathbf{e}_r + \mathbf{e}_z \right] [r_0 - f^*(z^*) + u^*] \right\} d\theta dz^*. \quad (2.14)
$$

Notice that the forth term in equation (4.32) is

$$
\mathbf{T}_2 ds_1 \big|_{\theta + d\theta} = T_2 \mathbf{e}_\theta (\theta + d\theta) ds_2. \quad (2.15)
$$

By expanding $\mathbf{e}_\theta (\theta + d\theta)$ with respect to $\theta$ yields

$$
\mathbf{e}_\theta (\theta + d\theta) = \mathbf{e}_\theta + \frac{d\mathbf{e}_\theta}{d\theta} d\theta = \mathbf{e}_\theta - \mathbf{e}_r d\theta. \quad (2.16)
$$

Thus equation (2.15) becomes

$$
\mathbf{T}_2 ds_1 \big|_{\theta + d\theta} = (T_2 \mathbf{e}_\theta - T_2 \mathbf{e}_r d\theta) \Lambda dz^*. \quad (2.17)
$$

Here, note that $T_2$ is independent of $\theta$. Then, the summation of the third and the forth terms in equation (4.32) results

$$
-T_2 ds_1 \big|_{\theta} + T_2 ds_1 \big|_{\theta + d\theta} = (-T_2 \mathbf{e}_\theta + T_2 \mathbf{e}_r d\theta) \Lambda dz^* = -T_2 \Lambda d\theta dz^* \mathbf{e}_r. \quad (2.18)
$$

Introducing equations (2.14) and (2.18) into equation (4.32), the total force acting to the tube element becomes

$$
d\mathbf{F} = \left\{ \frac{\partial}{\partial z^*} \left\{ \frac{T_1}{\Lambda} \left[ \left( -f^* + \frac{\partial u^*}{\partial z^*} \right) \mathbf{e}_r + \mathbf{e}_z \right] [r_0 - f^*(z^*) + u^*] \right\} \right\} - T_2 \Lambda \mathbf{e}_r \\
+ (T_2^* \mathbf{e}_r + T_2^* \mathbf{e}_z) [r_0 - f^*(z^*) + u^*] \Lambda d\theta dz^*, \quad (2.19)
$$
where $P_r^*$ and $P_z^*$ are the fluid reaction forces in the radial and axial directions, respectively. In addition to this force, there is also an axial tethering force, denoted by $T^*e_z$, per unit deformed area. Thus, the total force acting to the membrane element is given by

$$d\mathbf{F} = \left\{ \frac{\partial}{\partial z^*} \left[ \frac{\mathcal{L}_{1}}{\Lambda} \left[ r_0 - f^*(z^*) + u^* \right] \left( -f'' + \frac{\partial u^*}{\partial z^*} \right) \right] - T_2 \Lambda \\
+ P_r^*[r_0 - f^*(z^*) + u^*] \right\} e_z d\theta dz^* \\
+ \left\{ \frac{\partial}{\partial z^*} \left[ \frac{\mathcal{L}_{1}}{\Lambda} \left[ r_0 - f^*(z^*) + u^* \right] \right] + (P_z^* + T^*)[r_0 - f^*(z^*) + u^*] \right\} e_z d\theta dz^*.$$

(2.20)

According to the Newton's second law, where $F = ma$, which is given by

$$\rho_0 \frac{H}{\Lambda} \frac{\partial^2 u^*}{\partial t^2} d\theta dz^* e_r,$$

(2.21)

where $\rho_0$ is the mass density of the tube material. Equating the expressions (2.20) and (4.34), the following equations in component form are obtained.

$$\frac{\partial}{\partial z^*} \left[ \frac{\mathcal{L}_{1}}{\Lambda} \left[ r_0 - f^*(z^*) + u^* \right] \left( -f'' + \frac{\partial u^*}{\partial z^*} \right) \right] - T_2 \Lambda + \Lambda[r_0 - f^*(z^*) + u^*]P_r^*$$

$$= \rho_0 \frac{H}{\Lambda} \frac{\partial^2 u^*}{\partial t^2},$$

(2.22)

$$\frac{\partial}{\partial z^*} \left[ \frac{\mathcal{L}_{1}}{\Lambda} \left[ r_0 - f^*(z^*) + u^* \right] \right] + \Lambda[r_0 - f^*(z^*) + u^*](P_z^* + T^*) = 0.$$  

(2.23)

The equation (4.37) makes it possible to determine the tethering force $T^*$ in terms of the deformation.

Let $\mu \Sigma = \mu \Sigma(I_1, I_2)$ be the strain energy density function of the tube material, where $\mu$ is the shear modulus. For a homogeneous, incompressible, isotropic and elastic material, the stress tensor is given by Eringen (1962) as

$$\sigma_{kl} = \Pi \delta_{kl} + \mu(\Phi c_{kl}^{-1} + \Psi B_{kl}),$$

(2.24)

where $\Pi$ is the hydrostatic pressure, $\delta_{kl}$ is the Kronecker delta defined by

$$\delta_{kl} = \begin{cases} 1, & k = l, \\
0, & k \neq l, \end{cases}$$
and \(c_{kl}^{-1}\) is the Finger deformation tensor defined by
\[
c_{kl}^{-1} = F_{kK} F_{lK}, \quad F_{kK} = \frac{\partial x_k}{\partial X_K},
\]
(2.25)
where \(F_{kK}\) is the deformation gradient and \(x_k = x_k(X_K, t)\) is the motion. Other quantities \(B_{kl}, \Phi\) and \(\Psi\) are defined as
\[
B_{kl} = I_1 c_{kl}^{-1} - c_{km}^{-1} c_{ml}^{-1}, \quad \Phi = 2 \frac{\partial \Sigma}{\partial I_1}, \quad \Psi = 2 \frac{\partial \Sigma}{\partial I_2},
\]
(2.26)
where \(I_1, I_2\) and \(I_3 = 1\) (incompressibility) are the basic invariants of Finger deformation tensor defined by
\[
I_1 = c_{kk}^{-1}, \quad I_2 = \frac{1}{2} B_{kk}, \quad I_3 = \text{det}(c_{kl}^{-1}) = 1.
\]
(2.27)

By using (4.41) and incompressibility condition \(\lambda_1 \lambda_2 \lambda_3 = 1\) implies
\[
c_{11}^{-1} = \lambda_1^2, \quad c_{22}^{-1} = \lambda_2^2, \quad c_{33}^{-1} = \lambda_3^2 = \frac{1}{\lambda_1^2 \lambda_2^2}.
\]
(2.28)
Utilizing equation (4.44) into (4.42)\(_1\) yields
\[
B_{11} = \frac{1}{\lambda_2^2} + \lambda_1^2 \lambda_2^2, \quad B_{22} = \frac{1}{\lambda_1^2} + \lambda_1^2 \lambda_2^2, \quad B_{33} = \frac{1}{\lambda_3^2} + \frac{1}{\lambda_2^2}.
\]
(2.29)
Substituting equations (4.44) and (5.1) into (4.40), the components of stress tensor are obtained as
\[
t_{11} &= \Pi + \mu \left[ \Phi \lambda_1^2 + \Psi \left( \lambda_1^2 \lambda_2^2 + \frac{1}{\lambda_2^2} \right) \right], \\
t_{22} &= \Pi + \mu \left[ \Phi \lambda_2^2 + \Psi \left( \lambda_2^2 \lambda_1^2 + \frac{1}{\lambda_1^2} \right) \right], \\
t_{33} &= \Pi + \mu \left[ \Phi \frac{1}{\lambda_1^2 \lambda_2^2} + \Psi \left( \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right) \right], \\
t_{kl} &= 0, \quad k \neq l.
\]
(2.30)

For thin membrane, it is generally assumed that \(t_{33} \approx 0\). Using this approximation, it shows that
\[
\Pi = -\mu \left[ \Phi \frac{1}{\lambda_1^2 \lambda_2^2} + \Psi \left( \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right) \right].
\]
(2.31)
Substituting equation (5.3) into (5.2) yields

\[
\begin{align*}
t_{11} &= \mu \left[ \Phi \left( \lambda_1^2 \frac{1}{\lambda_1^2 \lambda_2^2} \right) + \Psi \left( \lambda_1^2 \lambda_2^2 - \frac{1}{\lambda_1^2} \right) \right] \\
&= \mu \lambda_1 \left[ \Phi \left( \lambda_1^2 \frac{1}{\lambda_1^2 \lambda_2^2} \right) + \Psi \left( \lambda_1^2 \lambda_2^2 - \frac{1}{\lambda_1^2} \right) \right], \\
t_{22} &= \mu \left[ \Phi \left( \lambda_2^2 \frac{1}{\lambda_2^2 \lambda_1^2} \right) + \Psi \left( \lambda_1^2 \lambda_2^2 - \frac{1}{\lambda_1^2} \right) \right] \\
&= \mu \lambda_2 \left[ \Phi \left( \lambda_2^2 \frac{1}{\lambda_2^2 \lambda_1^2} \right) + \Psi \left( \lambda_1^2 \lambda_2^2 - \frac{1}{\lambda_1^2} \right) \right]. 
\end{align*}
\]  

(2.32)

By using chain rule for differentiation, it can be expressed that

\[
\begin{align*}
\frac{\partial \Sigma}{\partial \lambda_1} &= \frac{\partial \Sigma}{\partial I_1} \frac{\partial I_1}{\partial \lambda_1} + \frac{\partial \Sigma}{\partial I_2} \frac{\partial I_2}{\partial \lambda_1}, \\
\frac{\partial \Sigma}{\partial \lambda_2} &= \frac{\partial \Sigma}{\partial I_1} \frac{\partial I_1}{\partial \lambda_2} + \frac{\partial \Sigma}{\partial I_2} \frac{\partial I_2}{\partial \lambda_2}. 
\end{align*}
\]  

(2.33)

By using equations (4.42) and (4.42) in equation (5.5), it can be shown that

\[
\frac{\partial \Sigma}{\partial I_1} = \frac{\Phi}{2}, \quad \frac{\partial \Sigma}{\partial I_2} = \frac{\Psi}{2}. 
\]  

(2.34)

From equations (4.43) and (4.44), it follows that

\[
I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2}, \quad I_2 = \lambda_1^2 \lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}. 
\]  

(2.35)

Differentiating equation (5.7) with respect to \( \lambda_1 \) and \( \lambda_2 \) yields

\[
\begin{align*}
\frac{\partial I_1}{\partial \lambda_1} &= 2 \lambda_1 - \frac{2}{\lambda_1^2 \lambda_2^2}, \quad \frac{\partial I_1}{\partial \lambda_2} = 2 \lambda_1 \lambda_2^2 - \frac{2}{\lambda_1^2}, \\
\frac{\partial I_2}{\partial \lambda_1} &= 2 \lambda_1 \lambda_2^2 - \frac{2}{\lambda_1^2}, \quad \frac{\partial I_2}{\partial \lambda_2} = 2 \lambda_2 - \frac{2}{\lambda_2^2 \lambda_1^2}.
\end{align*}
\]  

(2.36)

Using equations (5.6) and (5.8) into equation (5.5) results in

\[
\begin{align*}
\frac{\partial \Sigma}{\partial \lambda_1} &= \Phi \left( \lambda_1 - \frac{1}{\lambda_1^2 \lambda_2^2} \right) + \Psi \left( \lambda_1 \lambda_2^2 - \frac{1}{\lambda_1^2} \right), \\
\frac{\partial \Sigma}{\partial \lambda_2} &= \Phi \left( \lambda_2 - \frac{1}{\lambda_2^2 \lambda_1^2} \right) + \Psi \left( \lambda_1^2 \lambda_2 - \frac{1}{\lambda_2^2} \right). 
\end{align*}
\]  

(2.37)

Inserting equation (5.9) into equation (5.4) gives

\[
\begin{align*}
t_{11} &= \mu \lambda_1 \frac{\partial \Sigma}{\partial \lambda_1}, \quad t_{22} = \mu \lambda_2 \frac{\partial \Sigma}{\partial \lambda_2}. 
\end{align*}
\]  

(2.38)
Thus, the stress resultants $T_1$ and $T_2$ referred to the final configuration can be defined by

$$T_1 = h t_{11}, \quad T_2 = h t_{22}. \tag{2.39}$$

Introducing the incompressibility condition given in equation (4.31) and equation (6.2), the membrane forces $T_1$ and $T_2$ can be expressed in terms of the stretch ratios as

$$T_1 = \frac{\mu H}{\lambda_2} \frac{\partial \Sigma}{\partial \lambda_1}, \quad T_2 = \frac{\mu H}{\lambda_2} \frac{\partial \Sigma}{\partial \lambda_2}. \tag{2.40}$$

Substituting equation (6.1) into equation (4.35), the equation of motion of the tube in the radial direction takes the following form

$$\frac{\partial}{\partial z^*} \left\{ \frac{\mu H R_0}{\Lambda} \left( -f'' + \frac{\partial u^*}{\partial z^*} \right) \frac{\partial \Sigma}{\partial \lambda_1} \right\} - \frac{\mu H}{\lambda_2} \frac{\partial \Sigma}{\partial \lambda_2} + \Lambda [r_0 - f^*(z^*) + u^*] P^*_r - \rho_0 \frac{H}{\lambda_2} R_0 \frac{\partial^2 u^*}{\partial t^* 2} = 0. \tag{2.41}$$

The above equation is the equation of a stenosed tube.

2.3 Conclusion

In this chapter, the equation of a stenosed tube is derived. This equation (2.41) will be used in investigating the non-linear waves modulation in an inviscid fluid.
CHAPTER 3

EQUATIONS OF FLUID

3.1 Introduction

Generally, blood is treated as an incompressible non-Newtonian fluid. The concentration of red blood cells cause the blood to behave like a non-Newtonian fluid. Nevertheless, (Paquerot and Remoissenet, 1994) indicated that the blood viscosity is negligible in a number of applications. Therefore, for blood flow problems in large arteries, as a first approximation, blood can be considered as an incompressible inviscid fluid. The fluid reaction force density can be defined in terms of the stress tensor of fluid as

\[ P^*_r = -t_{rr} n_k \big|_{r^*=r_f} \]
\[ = -(t_{rr} n_r + t_{r2} n_z) \big|_{r^*=r_f} \]  \hspace{1cm} (3.1)

For inviscid fluids, the stress tensor \( t_{kl} \) can be expressed by

\[ t_{kl} = -\bar{P} \delta_{kl}, \]

where \( \bar{P} \) is the fluid pressure function. Setting \( P^*_r = \bar{P} \big|_{r^*=r_f} \), the fluid reaction force density can be given by

\[ P^*_r = \frac{P^*}{\Lambda}. \]  \hspace{1cm} (3.2)
From equation (4.29), one obtains \( n_r = \frac{1}{A} \). Thus, the equation of a stenosed tube (2.41) can be written as

\[
\frac{\partial}{\partial z^*} \left\{ \frac{\mu R_0}{A} \left( -f'^* + \frac{\partial u^*}{\partial z^*} \right) \frac{\partial \Sigma}{\partial \lambda_1} \right\} - \frac{\mu H}{\lambda_z} \frac{\partial \Sigma}{\partial \lambda_2} + [r_0 - f^*(z^*) + u^*]P^* - \rho_0 \frac{H}{\lambda_z} R_0 \frac{\partial^2 u^*}{\partial t^* \partial z^*} = 0. \tag{3.3}
\]

In this chapter, the non-linear wave modulation in a stenosed tube given in equation (3.3) filled with an inviscid fluid will be discussed.

### 3.2 Equations of an Inviscid Fluid

The equations of symmetrical motion of an inviscid fluid in the cylindrical polar coordinates \((r, \theta, z^*)\) may be given by

\[
\frac{\partial V_r^*}{\partial t^*} + V_r^* \frac{\partial V_r^*}{\partial r^*} + V_z^* \frac{\partial V_r^*}{\partial z^*} + \frac{1}{\rho_f} \frac{\partial \bar{P}}{\partial r^*} = 0, \tag{3.4}
\]

\[
\frac{\partial V_z^*}{\partial t^*} + V_r^* \frac{\partial V_z^*}{\partial r^*} + V_z^* \frac{\partial V_z^*}{\partial z^*} + \frac{1}{\rho_f} \frac{\partial \bar{P}}{\partial z^*} = 0, \tag{3.5}
\]

\[
\frac{\partial V_r^*}{\partial r^*} + \frac{V_r^*}{r} + \frac{\partial V_z^*}{\partial z^*} = 0 \quad \text{(incompressibility),} \tag{3.6}
\]

with the boundary conditions

\[
V_r^*|_{r^*=\tau_f} = \frac{\partial u^*}{\partial t^*} + \left[-f'^*(z^*) + \frac{\partial u^*}{\partial z^*}\right] V_z^*|_{r^*=\tau_f}, \quad \bar{P}|_{r^*=\tau_f} = P^*, \tag{3.7}
\]

where \(V_r^*\) and \(V_z^*\) are the components of the fluid velocity vector in the radial and axial directions respectively. \(\rho_f\) is the mass density of the fluid and \(r_f = r^* + u^*\) is the final inner radius of the tube.
3.3 Non-dimensionalization

At this stage it is convenient to introduce the following non-dimensional quantities

\[ t^* = \left( \frac{R_0}{c_0} \right) t, \quad z^* = R_0 z, \quad r^* = R_0 r, \quad u^* = R_0 u, \]
\[ f^* = R_0 f, \quad V_r^* = c_0 v, \quad V_z^* = c_0 q, \quad P^* = \rho_f c_0^2 \tilde{p}, \]
\[ m = \frac{\rho_0 H}{\rho_f R_0}, \quad c_0^2 = \frac{\mu H}{\rho_f R_0}, \]

(3.8)

where \( \lambda_0 = r_0/R_0 \) is the initial stretch ratio in the circumferential direction, \( r_0 \) is the radius at the origin of the coordinate system after finite static deformation, \( R_0 \) is the initial reference radius and \( c_0 \) is the Moens-Korteweg wave speed.

Introducing (3.8) into the equations (3.3)-(3.7), the following non-dimensional equations are obtained

\[ \tilde{p} = \frac{m \left( \lambda_0 - f(z) + u \right) \partial^2 u}{\lambda_2} + \frac{1}{\lambda_2} \frac{\partial \Sigma}{\partial z} \left\{ \frac{-f' + \partial u/\partial z}{\left[ 1 + (-f' + \partial u/\partial z)^2 \right]^{1/2}} \right\}, \]

(3.9)

\[ \frac{\partial v}{\partial t} + \frac{v}{r} + \frac{q}{z} + \frac{\partial \tilde{p}}{\partial r} = 0, \]

(3.10)

\[ \frac{\partial q}{\partial t} + \frac{v}{r} + \frac{q}{z} + \frac{\partial \tilde{p}}{\partial z} = 0, \]

(3.11)

\[ \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial q}{\partial z} = 0, \]

(3.12)

with the boundary conditions

\[ v|_{r=\lambda_0-f(z)+u} = \frac{\partial u}{\partial t} + \left( -f' + \frac{\partial u}{\partial z} \right) q|_{r=\lambda_0-f(z)+u}, \quad \tilde{p}|_{r=\lambda_0-f(z)+u} = \tilde{p}. \]

(3.13)

The equations (3.9)-(3.13) give sufficient relations to determine the field quantities \( u, v, q, \) and \( \tilde{p} \) completely.
CHAPTER 4

NON-LINEAR WAVE MODULATION

4.1 Introduction

In this section, we will examine the amplitude modulation of weakly non-linear waves in a fluid-filled thin elastic tube with a stenosis whose non-dimensional governing equations are given in equations (3.9)-(3.13). Considering the dispersion relation of the linearized field equations and the nature of the problem of concern, which is a boundary-value problem, the following stretched coordinates is introduced:

\[ \xi = \epsilon(z - \lambda t), \quad \tau = \epsilon^2 z, \]  \hspace{1cm} (4.1)

where \( \epsilon \) is a small parameter measuring the weakness of nonlinearity and \( \lambda \) is a constant to be determined from the solution. Solving \( z \) in terms of \( \tau \), we get

\[ z = \epsilon^{-2} \tau. \]  \hspace{1cm} (4.2)

Introducing (4.2) into the expression of \( f(z) \), we obtain

\[ f(\epsilon^{-2} \tau) = \hat{h}(\tau). \]  \hspace{1cm} (4.3)

In order to take the effect of stenosis into account, \( f(z) \) must be of order of \( \epsilon^4 \). For the present work, we shall assume that \( \hat{h}(\tau) \) have the following form

\[ \hat{h}(\tau) = \epsilon^2 h(\tau). \]  \hspace{1cm} (4.4)

Introducing the following differential relations

\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \epsilon \lambda \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} + \epsilon \frac{\partial}{\partial \xi} + \epsilon^2 \frac{\partial}{\partial \tau}, \]  \hspace{1cm} (4.5)
and assuming that the field quantities can be expanding into asymptotic series of $\epsilon$ as

$$
\begin{align*}
    u &= \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \ldots, \\
    v &= \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \ldots, \\
    q &= \epsilon q_1 + \epsilon^2 q_2 + \epsilon^3 q_3 + \ldots, \\
    \tilde{p} &= \tilde{p}_0 + \epsilon \tilde{p}_1 + \epsilon^2 \tilde{p}_2 + \epsilon^3 \tilde{p}_3 + \ldots, \\
    p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + \ldots, \\
    h(\tau) &= \epsilon^2 h_1(\tau) + \epsilon^3 h_2(\tau) + \ldots
\end{align*}
$$

where $u, v, q, \tilde{p}$ and $p$ are functions of the slow variables $(\xi, \tau)$ as well as the fast variables $(z, t)$.

Introducing (4.5) and (4.6) into the equations (3.9)-(3.13), the following sets of differential equations are obtained

### 4.2 $O(\epsilon)$ equations

$$
\begin{align*}
    p_1 &= \frac{m}{\lambda_0 \lambda_z} \frac{\partial^2 u_1}{\partial t^2} - \alpha_0 \frac{\partial^2 u_1}{\partial z^2} + \beta_1 u_1, \\
    \frac{\partial u_1}{\partial t} + \frac{\partial \tilde{p}_1}{\partial \tau} &= 0, \\
    \frac{\partial q_1}{\partial t} + \frac{\partial \tilde{p}_1}{\partial z} &= 0, \\
    \frac{\partial v_1}{\partial \tau} + \frac{v_1}{r} + \frac{\partial q_1}{\partial z} &= 0
\end{align*}
$$

(4.7)

and the boundary conditions

$$
\begin{align*}
    v_1|_{r=\lambda_0} &= \frac{\partial u_1}{\partial t}, \\
    \tilde{p}_1|_{r=\lambda_0} &= p_1
\end{align*}
$$

(4.8)
4.3 \(O(\varepsilon^2)\) equations

\[
p_2 = \frac{m}{\lambda_0 \lambda_z} \frac{\partial^2 u_2}{\partial t^2} - \frac{\alpha_0}{\lambda_z} \frac{\partial^2 u_2}{\partial z^2} + \beta_1 (u_2 - h_1),
\]
\[
- \alpha_1 \left( \frac{\partial u_1}{\partial z} \right)^2 - \left( 2\alpha_1 - \frac{\alpha_0}{\lambda_0} \right) \frac{u_1}{\lambda_0} \frac{\partial^2 u_1}{\partial z^2} + \beta_2 u_1^2,
\]
\[
\frac{\partial v_2}{\partial t} - \lambda \frac{\partial v_1}{\partial \xi} + v_1 \frac{\partial v_1}{\partial r} + q_1 \frac{\partial v_1}{\partial z} + \frac{\partial p_2}{\partial r} = 0,
\]
\[
\frac{\partial q_2}{\partial t} - \lambda \frac{\partial q_1}{\partial \xi} + v_1 \frac{\partial q_1}{\partial r} + q_1 \frac{\partial q_1}{\partial z} + \frac{\partial p_2}{\partial z} + \frac{\partial p_1}{\partial \xi} = 0,
\]
\[
\frac{\partial v_2}{\partial r} + \frac{v_2}{r} + \frac{\partial q_2}{\partial z} + \frac{\partial q_2}{\partial \xi} = 0,
\]
(4.9)

and the boundary conditions

\[
\left[ u_1 \frac{\partial v_1}{\partial r} + v_2 \right]_{r=\lambda_0} = \frac{\partial u_2}{\partial t} - \lambda \frac{\partial u_1}{\partial \xi} + \frac{\partial u_1}{\partial z} q_1 \bigg|_{r=\lambda_0},
\]
\[
\left[ u_1 \frac{\partial p_1}{\partial r} + \bar{p}_2 \right]_{r=\lambda_0} = p_2 (4.10)
\]
4.4 $O(\epsilon^3)$ equations

\[ p_3 = \frac{m}{\lambda_0 \lambda_z} \frac{\partial^2 u_3}{\partial t^2} - \frac{m}{\lambda_0} \frac{\partial^2 u_3}{\partial z^2} - \frac{2m\lambda}{\lambda_0 \lambda_z} \frac{\partial^2 u_2}{\partial z \partial t} - \frac{2m}{\lambda_0} \frac{\partial^2 u_2}{\partial \xi \partial z} - \alpha_0 \left( \frac{\partial^2 u_1}{\partial \xi^2} + 2 \frac{\partial u_1}{\partial \tau} \right) + \frac{m}{\lambda_0 \lambda_z} \frac{\partial^2 u_1}{\partial \xi \partial t} + \frac{m}{\lambda_0^2} \left( \frac{\partial u_2}{\partial \xi} + \frac{\partial u_2}{\partial t} \right) \right] - \alpha_0 \left( \frac{\partial^2 u_1}{\partial \xi^2} + 2 \frac{\partial u_1}{\partial \tau} \right) + \frac{m}{\lambda_0 \lambda_z} \frac{\partial^2 u_1}{\partial \xi \partial t} + \frac{m}{\lambda_0^2} \left( \frac{\partial u_2}{\partial \xi} + \frac{\partial u_2}{\partial t} \right) \right] - \left( 2 \alpha_1 - \frac{\alpha_0}{\lambda_0} \right) \frac{\partial u_1}{\partial \xi} \left( \frac{\partial u_1}{\partial \xi} \right) \right] - \left( \alpha_2 - \frac{2 \alpha_1}{\lambda_0} + \frac{\alpha_0}{\lambda_0^2} \right) \frac{\partial u_1}{\partial \xi} \left( \frac{\partial u_1}{\partial \xi} \right) \right] - \left( \gamma_1 - \frac{\alpha_0}{2} \right) \left( \frac{\partial u_1}{\partial \xi} \right) \left( \frac{\partial^2 u_1}{\partial \xi^2} + 2 \frac{\partial^2 u_1}{\partial \xi \partial t} + \beta_3 u_1^3 \right),

\[ \frac{\partial v_3}{\partial t} + \lambda \frac{\partial v_2}{\partial \xi} + v_3 \frac{\partial v_2}{\partial r} + v_2 \frac{\partial v_1}{\partial r} + q_1 \frac{\partial v_2}{\partial z} + q_1 \frac{\partial v_1}{\partial \xi} \right] + q_2 \frac{\partial v_1}{\partial z} + q_3 \frac{\partial v_3}{\partial \xi} = 0,

\[ \frac{\partial q_3}{\partial r} + \frac{\partial q_3}{\partial z} + \frac{\partial q_3}{\partial \xi} + \frac{\partial q_1}{\partial \tau} = 0,

\text{and the boundary conditions}

\[ \left[ \frac{1}{2} u_1^2 \frac{\partial^2 v_1}{\partial t^2} + \left( u_2 - h_1 \right) \frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_2}{\partial r} + v_3 \right] \bigg|_{r = \lambda_0} = \frac{\partial u_3}{\partial t} - \lambda \frac{\partial u_2}{\partial \xi}

\[ + \frac{\partial v_1}{\partial z} \left[ \frac{\partial q_3}{\partial r} + q_2 \right] \bigg|_{r = \lambda_0} + \left[ \frac{\partial u_2}{\partial z} + \frac{\partial u_1}{\partial \xi} \right] q_1 \bigg|_{r = \lambda_0},

\[ \left[ \frac{1}{2} u_1^2 \frac{\partial^2 p_3}{\partial t^2} + \left( u_2 - h_1 \right) \frac{\partial p_3}{\partial t} + u_1 \frac{\partial p_3}{\partial r} + p_3 \right] \bigg|_{r = \lambda_0} = p_3. \]
Here the coefficients of $\alpha_0$, $\alpha_1$, $\alpha_2$, $\beta_0$, $\beta_1$, $\beta_2$, $\beta_3$ and $\gamma_1$ are defined by

$$
\alpha_0 = \frac{1}{\lambda_0} \frac{\partial \Sigma}{\partial \lambda_z}, \quad \alpha_1 = \frac{1}{2\lambda_0} \frac{\partial^2 \Sigma}{\partial \lambda_0 \partial \lambda_z}, \quad \alpha_2 = \frac{1}{2\lambda_0} \frac{\partial^3 \Sigma}{\partial \lambda_0^2 \partial \lambda_z},
$$

$$
\beta_0 = \frac{1}{\lambda_0 \lambda_z} \frac{\partial \Sigma}{\partial \lambda_0}, \quad \beta_1 = \beta_0 \lambda_0 \frac{\partial \lambda_0}{\partial \lambda_z}, \quad \beta_2 = \frac{1}{\lambda_0} \frac{\partial^2 \Sigma}{\partial \lambda_0 \partial \lambda_z} \frac{\beta_1}{\lambda_0},
$$

$$
\beta_3 = \frac{1}{\lambda_0} \frac{\partial^4 \Sigma}{\partial \lambda_0 \partial \lambda_z^3} - \frac{\beta_2}{\lambda_0}, \quad \gamma_1 = \frac{\lambda_z}{2\lambda_0} \frac{\partial^2 \Sigma}{\partial \lambda_0^2}, \quad \gamma_1 = \frac{\lambda_z}{2\lambda_0} \frac{\partial^2 \Sigma}{\partial \lambda_0^2} \frac{\beta_1}{\lambda_0}.
$$

Equation (4.13) are defined through series expansion of the stretch ratios $\lambda_1$ and $\lambda_2$, which read

$$
\lambda_1 = \lambda_0 \left[ 1 + \left( \varepsilon \frac{\partial u_1}{\partial z} + \varepsilon^2 \left[ \frac{\partial u_2}{\partial z} + \frac{\partial u_1}{\partial \xi} \right] + \varepsilon^3 \left[ \frac{\partial u_3}{\partial z} + \frac{\partial u_2}{\partial \xi} + \frac{\partial u_1}{\partial \tau} \right] \right) \right]^{1/2},
$$

$$
\lambda_2 = \lambda_0 + \varepsilon u_1 + \varepsilon^2 [u_2 - h_1(\tau)] + \varepsilon^3 [u_3 - h_2(\tau)].
$$

### 4.5 The Solution of $O(\varepsilon)$ equations

Seeking the following type of solution to the differential equations (4.7):

$$
\begin{align*}
&u_1 = U_1 e^{i\theta} + \text{c.c.}, & u_1 = V_1 e^{i\theta} + \text{c.c.}, & q_1 = Q_1 e^{i\theta} + \text{c.c.,} \\
&\bar{p}_1 = \bar{P}_1 e^{i\theta} + \text{c.c.,} & p_1 = P_1 e^{i\theta} + \text{c.c.,}
\end{align*}
$$

(4.15)

where $U_1$ and $P_1$ are amplitude functions of the slow variables $(\xi, \tau)$, $V_1$, $\bar{P}_1$ and $Q_1$ are amplitude functions which depend on $r$ as well as the slow variables $(r; \xi, \tau)$, $\theta = \omega t - kz$ is the phasor where $\omega$ is the angular frequency, $k$ is the wave number and c.c is the complex conjugate of the corresponding expressions. Thus, the solution satisfying the differential equations (4.7) and the boundary conditions (4.8) are given as follows:

$$
\begin{align*}
&U_1 = U(\xi, \tau), & P_1 = \frac{\omega^2}{k} f_0(k\lambda_0) U, & V_1 = i\omega f_1(kr) U, \\
&Q_1 = \omega f_0(kr) U, & P_1 = \frac{\omega^2}{k} f_0(kr) U,
\end{align*}
$$

(4.16)

provided that the following dispersion relation holds true:

$$
\frac{\omega^2 f_0}{k} + \frac{m\omega^2}{\lambda_0 \lambda_z} - \alpha_0 k^2 - \beta_1 = 0.
$$

(4.17)
Here for simplicity, we have defined the following functions:

\[ f_n(kr) = \frac{I_n(kr)}{I_1(k\lambda_\theta)}, \quad (n = 0, 1, 2, \ldots), \quad f_0 = f_0(k\lambda_\theta), \]

where \( I_n(r) \) referred as modified Bessel's function of order \( n \) and \( U(\xi, r) \) is an unknown function whose governing equation will be obtained later. The group velocity, \( \lambda \) is defined as

\[ \lambda = \frac{\omega^2 \lambda_\theta [f_0^2 - 1] + 2\alpha_0 k^2}{2\omega [f_0 + \frac{mk}{\lambda_\theta \lambda_z}].} \]

4.6 The Solution of \( O(\varepsilon^2) \) equations

Introducing the solutions (4.15)-4.16 into equation (4.9) and (4.10) gives

\[
\frac{\partial \psi_2}{\partial t} + \frac{\partial \bar{\psi}_2}{\partial r} - i\omega \lambda f_1(kr) \left( \frac{\partial U}{\partial \xi} \right) e^{i\theta} + \frac{\omega^2 r}{f_1^2(kr)} U^2 e^{2i\theta} \\
+ 2\omega^2 \left[ 2k f_0(kr) f_1(kr) - \frac{f_1^2(kr)}{r} \right] |U|^2 + c.c = 0,
\]

\[
\frac{\partial q_2}{\partial t} + \frac{\partial \bar{q}_2}{\partial z} + \omega \left( \frac{\omega}{k} - \lambda \right) f_0(kr) \left( \frac{\partial U}{\partial \xi} \right) e^{i\theta} \\
+ i\omega^2 k \left[ f_1^2(kr) - f_0^2(kr) \right] U^2 e^{2i\theta} + c.c = 0,
\]

\[
\frac{\partial v_2}{\partial t} + \frac{\partial \bar{v}_2}{\partial r} + \frac{\partial q_2}{\partial z} + \omega f_0(kr) \left( \frac{\partial U}{\partial \xi} \right) e^{i\theta} + c.c = 0,
\]

and the boundary conditions

\[
\frac{\partial \psi_2}{\partial t} - v_2 |_{r = \lambda_\theta} = i\omega \left[ 2k f_0 - \frac{1}{\lambda_\theta} \right] U^2 e^{2i\theta} + \lambda \frac{\partial U}{\partial \xi} e^{i\theta} + c.c, \quad p_2 - \bar{p}_2 |_{r = \lambda_\theta} = \omega^2 U^2 e^{2i\theta} + 2\omega^2 |U|^2 + c.c.,
\]

where \( |U|^2 = UU^*, U^* \) is the complex conjugate of \( U \) and noticed that \( f_1(k\lambda_\theta) = 1. \)
Seeking the following type of solutions

\begin{align*}
    u_2 &= U_2^{(0)} + \left( \sum_{l=1}^{2} U_2^{(l)}(\xi, \tau)e^{il\theta} + c.c. \right), \\
    v_2 &= V_2^{(0)} + \left( \sum_{l=1}^{2} V_2^{(l)}(r; \xi, \tau)e^{il\theta} + c.c. \right), \\
    \varphi_2 &= Q_2^{(0)} + \left( \sum_{l=1}^{2} Q_2^{(l)}(r; \xi, \tau)e^{il\theta} + c.c. \right), \\
    p_2 &= P_2^{(0)} + \left( \sum_{l=1}^{2} P_2^{(l)}(\xi, \tau)e^{il\theta} + c.c. \right),
\end{align*}

(4.22)

to equations (4.20) and (4.21), we obtain the following set of differential equations and the boundary conditions

\begin{align*}
    P_2^{(0)} &= \beta_1 \left[ U_2^{(0)} - h_2 \right] + 2 \left( \frac{\beta_1}{\lambda_0} - \frac{\omega^2 f_0}{k\lambda_0} + \alpha_1k^2 + \beta_2 \right) |U|^2, \\
    \frac{\partial \tilde{P}_2^{(0)}}{\partial r} + 2\omega^2 \left[ 2kf_0(kr)f_1(kr) - \frac{f_1^2(kr)}{\tau} \right] |U|^2 &= 0, \\
    \frac{\partial V_2^{(0)}}{\partial r} + \frac{V_2^{(0)}}{r} &= 0,
\end{align*}

(4.23)

and the boundary conditions

\begin{align*}
    V_2^{(0)}|_{r=\lambda_0} = 0, \quad P_2^{(0)} - \tilde{P}_2^{(0)}|_{r=\lambda_0} = 2\omega^2|U|^2.
\end{align*}

(4.24)

The equations and the boundary conditions resulting from the coefficient of \(e^{i\theta}\) become

\begin{align*}
    P_2^{(1)} &= \frac{\omega^2}{k} f_0 U_2^{(1)} + 2i \left[ \alpha_0 k - \frac{m\omega}{\lambda_0} \lambda \right] \frac{\partial U}{\partial \xi}, \\
    i\omega V_2^{(1)} + \frac{\partial \tilde{P}_2^{(1)}}{\partial r} - i\omega f_1(kr) \frac{\partial U}{\partial \xi} &= 0, \\
    i\omega Q_2^{(1)} - ik\tilde{P}_2^{(1)} + \omega \left[ \frac{\omega}{k} - \lambda \right] f_0(kr) \frac{\partial U}{\partial \xi} &= 0, \\
    \frac{\partial V_2^{(1)}}{\partial r} + \frac{V_2^{(1)}}{r} - ikQ_2^{(1)} + \omega f_0(kr) \frac{\partial U}{\partial \xi} &= 0,
\end{align*}

(4.25)

and the boundary conditions

\begin{align*}
    i\omega U_2^{(1)} - V_2^{(1)}|_{r=\lambda_0} = \lambda \frac{\partial U}{\partial \xi}, \quad P_2^{(1)} - \tilde{P}_2^{(1)}|_{r=\lambda_0} = 0.
\end{align*}

(4.26)
REFERENCES


Demiray, H. (2002). Nonlinear waves in a prestressed elastic tube filled


