Soliton Cellular Automata constructed from a $U_q(D_n^{(1)})$ -Crystal $B^{n,1}$ and Kirillov-Reshetikhin type bijection for $U_q(E_6^{(1)})$ -Crystal $B^{6,1}$

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ΒY

MAHATHIR BIN MOHAMAD

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Abstract

In part 1 we study a class of cellular automata associated with the Kirillov-Reshetikhin crystal $B^{n,1}$ of type $D_n^{(1)}$. They have a commuting family of time evolutions and solitons of length l are labeled by $U_q(A_{n-1}^{(1)})$ -crystal $B_A^{2,l}$. The scattering rule of two solitons of lengths l_1 and l_2 $(l_1 > l_2)$ including the phase shift is identified with the combinatorial R-matrix for the $U_q(A_{n-1}^{(1)})$ -crystal $B_A^{2,l} \otimes B_A^{2,l}$. In part 2 we consider the Kirrilov-Reshetikhin crystal $B^{6,1}$ for the exceptional affine type $E_6^{(1)}$. We will give a conjecture on a statistic-preserving bijection between the highest weight paths consisting of $B^{6,1}$ and the corresponding rigged configuration. The algorithm only uses the structure of the crystal graph, hence could also be applied for other exceptional types. Our $B^{6,1}$ has a different algorithm compared our $B^{1,1}$ because we must consider the element ϕ , unique element in the highest weight crystal of weight 0, in the crystal graph. We will give many examples supporting the conjecture.

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Part I Scattering Rules in Soliton Cellular Automata Associated with $U_q(D_n^{(1)})$ -Crystal $B^{n,1}$

1 Introduction

The box-ball system [33, 32] well-known as soliton cellular automata is a dynamical system of balls in a one dimensional array of boxes. The discrete KdV equation through a limiting procedure called ultradiscretization [36] was used to show the solitonic character like the KdV solitons. The rules for soliton interactions and factorization property of scattering matrices (Yang-Baxter equation) are justified by means of inverse ultradiscretization [35]. In [35] it is shown that the dynamical systems of soliton cellular automaton is described by an ultra-discrete equation obtained from extended Toda molecule equation. Later it was studied by [3] that the scattering of two solitons including the phase shift is described by isomorphism from the tensor product of two affine crystals for the quantum enveloping algebra $U_q(A_{n-1}^{(1)})$ to the other order of the tensor product. The object they used is called combinatorial *R*-matrix [13]. The combinatorial *R*-matrix has an amazing property: it satisfies the Yang-Baxter equation, which assures that the scattering of three solitions does not depend on the order of scattering of the two solitons.



The new soliton cellular automata were constructed in [8] corresponding to $U_q(\mathfrak{g}_n)$ where $\mathfrak{g}_n = A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}$ and their internal degree of freedom was labeled by crystals of the smaller algebra $U_q(\mathfrak{g}_{n-1})$. Then [7] studied the scattering rule of two solitons when they collide each other. They found that the scattering rule for affine crystals corresponding to $U_q(\mathfrak{g}_n)$ can be described by combinatorial Rmatrix of the smaller algebra $U_q(\mathfrak{g}_{n-1})$. The affine crystal they used is called Kirillov-Reshetikhin (KR) crystal denoted by $B^{1,1}$. (The KR crystal is parameterized by two integers. The first index corresponds to a node of the Dynkin diagram of the affine algebra except 0 and the second a positive integer.) A generalization to the KR crystal $B^{k,1}$ for $\mathfrak{g} = A_{n-1}^{(1)}$ was studied in [37] and their internal degree of freedom is given by the product of $U_q(A_{k-1}^{(1)})$ -crystal $B^{k-1,l}$ and $U_q(A_{n-k-1}^{(1)})$ -crystal $B^{1,l}$. A case for the exceptional algebra $\mathfrak{g} = D_4^{(3)}$ was also treated in [38]. In[38], it is shown that the scattering rule for the crystal type $U_q(D_4^{(3)})$ is identified with the combinatorial R-matrix for $U_q(A_1^{(1)})$ -crystals and phase shifts are given by 3-times of those in the well-known box ball system.

These results can be summarized and one might find the following conjectural properties for the solitons and their scatterings of the soliton cellular automaton constructed from the KR crystal $B^{k,1}$ of the quantum affine algebra $U_q(\mathfrak{g})$. Let G be the Dynkin diagram of the corresponding finite-dimensional simple Lie algebra of \mathfrak{g} . Let \check{G} be the Dynkin diagram obtained by removing the node k from G and let j be the node of \check{G} that is connected to k in G. Let $\check{\mathfrak{g}}$ be the corresponding affine algebra.

- The internal degree of freedom of soliton of length l is described by the $U_q(\check{\mathfrak{g}})$ -crystal $B^{j,l}$.
- \bullet The exchange of the internal degree of freedom by the scattering of solitons of length
- l_1 and l_2 $(l_1 > l_2)$ is given by the crystal isomorphism $B^{j,l_1} \otimes B^{j,l_2} \xrightarrow{\sim} B^{j,l_2} \otimes B^{j,l_1}$.
- The phase shift of the scattering is described by the corresponding H function.

This property allows us to calculate the combinatorial R matrix for $B^{j,l_1} \otimes B^{j,l_2}$ just by observing the scattering of solitons in the corresponding cellular automaton. If \check{G} is not connected, then one needs to consider the tensor product as seen in [37]. Although this conjecture seems reasonable, the rigorous proof is yet to be given.

The purpose of this paper is to add another affirmative example to this conjecture. We take $\mathfrak{g} = D_n^{(1)}, k = n$. The corresponding node is a spin node and the KR crystal is $B^{n,1}$. According to the above conjecture, we have $\check{\mathfrak{g}} = A_{n-1}^{(1)}, j = n - 2$. In the crystal theory, there is a notion of the dual crystal. The dual B^{\vee} of a crystal B is defined by setting $(e_ib)^{\vee} = f_ib^{\vee}, (f_ib)^{\vee} = e_ib^{\vee}$. Since we know $(B_1 \otimes B_2)^{\vee} = B_2^{\vee} \otimes B_1^{\vee}$ and $(B^{j,l})^{\vee} = B^{n-j,l}$ for the KR crystal of type $A_{n-1}^{(1)}$, we can expect the following property on our soliton cellular automaton.

• The internal degree of freedom of a soliton of length l is described by the $U_q(A_{n-1}^{(1)})$ crystal $B_A^{2,l}$.

• The exchange of the internal degree of freedom by the scattering of solitons of length

 l_1 and l_2 $(l_1 > l_2)$ is given by the crystal isomorphism $B_A^{2,l_2} \otimes B_A^{2,l_1} \xrightarrow{\sim} B_A^{2,l_1} \otimes B_A^{2,l_2}$.

• The phase shift of the scattering is described by the corresponding H function.

We check these properties in this paper, thereby obtain our main theorem (Theorem 3.16).

The paper is organized as follows. In Sec. 2, we recapitulate necessary facts from the crystal theory. In Sec. 3, we construct conserved quantities. The main theorem is given in Sec. 3, where the scattering of solitons is studied.

2 Preliminaries

In this section we review some basic definitions and facts about crystals for the $U'_q(D_n^{(1)})$ -crystal $B^{n,l}$ in Section 2.1. In order to describe the crystal graphs for the finite-dimensional modules of quantum groups of classical type, Kashiwara and Nakashima introduced the analogue of semi-standard tableaux, called Kashiwara-Nakashima (KN) tableaux [16].



2.1 Crystal $B^{n,1}$

Crystal theory was introduced by Kashiwara [12] which provides a combinatorial way to study the representation theory of the quantum algebra $U_q(\mathfrak{g})$. In this paper $\mathfrak{g} = D_n^{(1)}$ is the corresponding quantum algebra. Let P be the weight lattice, $\{\alpha_i\}_{0 \leq i \leq n}$ the simple roots, and $\{\Lambda_i\}_{0 \leq i \leq n}$ the fundamental weights of $D_n^{(1)}$. Let $\overline{\Lambda}_i$ denote the classical part of Λ_i . The crystal B is a finite set with weight decomposition $B = \bigsqcup_{\lambda \in P} B_{\lambda}$. The Kashiwara operators e_i, f_i $(i = 0, 1, \dots, n)$ act on B as

$$e_i: B_{\lambda} \longrightarrow B_{\lambda + \alpha_i} \sqcup \{0\}, \quad f_i: B_{\lambda} \longrightarrow B_{\lambda - \alpha_i} \sqcup \{0\}.$$

These operators are nilpotent. By definition, we have $f_i b = b'$ if and only if $b = e_i b'$. Drawing $b \xrightarrow{i} b'$ in such case, B is endowed with the structure of colored oriented graph called *crystal graph*.

Let $\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$ orthonormal basis of the weight space of D_n . The simple roots and classical parts of fundamental weights for $D_n^{(1)}$ are expressed as

$$\alpha_0 = \delta - \epsilon_1 - \epsilon_2, \quad \alpha_n = \epsilon_{n-1} + \epsilon_n, \quad \alpha_i = \epsilon_i - \epsilon_{i+1}, \quad \text{for } i = 1, 2, ..., n - 1,$$

$$\bar{\Lambda}_{n-1} = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{n-1} - \epsilon_n), \quad \bar{\Lambda}_n = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{n-1} + \epsilon_n).$$

$$\bar{\Lambda}_i = \epsilon_1 + \epsilon_2 + \dots + \epsilon_i \quad \text{for } i = 1, 2, ..., n - 2,$$

We explain the Kirillov-Reshetikhin crystal $B^{n,l}, \ l \in \mathbb{Z}_{>0}$. We set

 $\mathfrak{A} = \{1, 2, \cdots, n-1, n, \overline{n}, \overline{n-1}, \cdots, \overline{2}, \overline{1}\}.$ The set of letters order on

$$\mathfrak{A}: 1 \prec 2 \prec \cdots \prec n-1 \prec \frac{n}{\overline{n}} \prec \overline{n-1} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

where there is no order between n and \bar{n} . Then the crystal $B^{n,l}$ is given by

$$B^{n,1} = \left\{ \begin{array}{c|c} i_n \\ \vdots \\ i_2 \\ \hline i_1 \\ \hline \end{array} \right| \begin{array}{c} i_p \in \mathfrak{A}, i_1 \prec i_2 \prec \cdots \prec i_n, \\ a, \bar{a} \text{ does not coexist for any } a = 1, 2, \dots, n. \\ \text{There are even number of barred letters.} \end{array} \right\},$$
(2.1)

$$B^{n,l} = \left\{ \begin{bmatrix} c_1 & c_2 \\ & & \\ & & \\ \end{bmatrix} : \cdots \begin{bmatrix} c_l \\ & & \\$$

The weight of $b \in B^{n,1}$ is given by $wt \ b = \frac{1}{2} \sum_{j=1}^{n} \eta_j \epsilon_j$ where

$$\eta_j = \begin{cases} +1 & \text{if } j \text{ exist in } b, \\ \\ -1 & \text{if } \overline{j} \text{ exist in } b \end{cases}$$

 $\in B^{n,l}$ is given by $wt \ b = \sum_{j=1}^{l} wt \ c_j$. c_1 c_2 c_l and that of b =AAN TUNKU

Crystal structure on $B^{n,1}$ 2.2

For i = 0, 1, ..., n

$$b = \begin{cases} b' & \text{if } wt \ b' = wt \ b + \alpha_i \pmod{\mathbb{Z}\delta}, \\\\ 0 & \text{if such } b' \text{ does not exist in } B^{n,1}, \end{cases}$$

$$f_i b = \begin{cases} b'' & \text{if } wt \ b'' = wt \ b - \alpha_i \pmod{\mathbb{Z}\delta}, \\ 0 & \text{if such } b'' \text{ does not exist in } B^{n,1} \end{cases}$$

 $B^{n,1}$ is the crystal base [12] of the spin representation of the quantum affine algebra $U_q'(D_n^{(1)}).$

Example 2.1 When n = 4, the crystal graph of $B^{4,1}$ is depicted as follows.



The crystal graph of $B^{1,1}$ is the same as above by interchanging the colors as $1 \leftrightarrow 4$.

Example 2.2 When n = 5, the crystal graph of $B^{5,1}$ is depicted as follows.



We give the correspondence between the numbers in the crystal graph with our representation of crystal elements.

$$\begin{bmatrix}
5 & 4 & 3 & 2 & 1 & 3 \\
4 & 5 & 5 & 5 & 5 & 5 \\
2 & 3 & 3 & 4 & 4 & 5 \\
2 & 2 & 2 & 3 & 3 & 2 \\
1 & 1 & 1 & 1 & 2 & 1
\end{bmatrix}$$



Example 2.3 When n = 6, the crystal graph of $B^{6,1}$ is depicted as follows.



We give the correspondence between the numbers in the crystal graph with our representation of crystal elements.

Example 2.4 Consider the case n = 4.

Let
$$c_0 = \begin{pmatrix} 4 & \bar{3} & \bar{2} & \bar{1} & \bar{1} & \bar{1} \\ 3 & c^{(1)} = & \frac{\bar{4}}{2} & c^{(2)} = & \frac{\bar{4}}{3} & c^{(3)} = & \frac{\bar{4}}{3} & c^{(4)} = & \frac{\bar{3}}{4} & c^{(5)} = & \frac{\bar{1}}{2} \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 \end{pmatrix}$$

$$e_2(c^{(2)}) = c^{(1)}, \qquad f_2(c^{(2)}) = 0, \qquad f_0(c^{(2)}) = 0.$$

Example 2.5 Consider the case n = 5.

$$\begin{aligned} & 5 & \bar{4} & \bar{3} & \bar{2} & \bar{1} & \bar{3} \\ & 4 & \bar{5} & \bar{5} & \bar{5} & \bar{5} & \bar{5} \\ & 4 & \bar{5} & \bar{5} & \bar{5} & \bar{5} & \bar{5} \\ & 2 & 2 & 2 & 4 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 2 & 1 \end{aligned}$$

$$e_2(\tilde{c}^{(2)}) = 0, \qquad f_2(\tilde{c}^{(2)}) = \tilde{c}^{(3)}, \qquad f_0(\tilde{c}^{(2)}) = 0. \end{aligned}$$

2.3 Crystal structure on $B^{n,l}$

Let $b \in B^{n,l}$.

$$b = \begin{bmatrix} c_1 & c_2 \\ & & \ddots \end{bmatrix} \cdots \begin{bmatrix} c_l \\ & & \\ & \\ & & \\$$

The actions of e_i , f_i for $i \neq 0$ can be calculated by using the rule called **signature rule**. For $b \in B^{n,l}$ we associate an element $c_l \otimes c_{l-1} \otimes \cdots \otimes c_2 \otimes c_1$ of the tensor product $(B^{n,1})^{\otimes l}$ to find the indices j, j' such that

$$e_i(c_l \otimes c_{l-1} \otimes \cdots \otimes c_2 \otimes c_1) = c_l \otimes c_{l-1} \otimes \cdots \otimes e_i c_j \otimes \cdots \otimes c_2 \otimes c_1$$
(2.4)

$$f_i(c_l \otimes c_{l-1} \otimes \cdots \otimes c_2 \otimes c_1) = c_l \otimes c_{l-1} \otimes \cdots \otimes f_i c_{j'} \otimes \cdots \otimes c_2 \otimes c_1$$
(2.5)

With this element we associate an *i*-signature : $\underbrace{-\dots-}_{\varepsilon_i(c_l)} \underbrace{+\dots+}_{\varphi_i(c_l)} \underbrace{-\dots-}_{\varepsilon_i(c_{l-1})} \underbrace{+\dots+}_{\varphi_i(c_{l-1})} \dots \underbrace{-\dots-}_{\varepsilon_i(c_1)} \underbrace{+\dots+}_{\varphi_i(c_1)}$

We then reduce the signature by deleting the adjacent +- pair successively. Eventually we obtain a reduced signature of the following form.

 $--\cdots ++\cdots +$

Then the action e_i (resp. f_i) corresponds to changing the rightmost – to + (resp. leftmost + to –). If there is no – (resp. +) in the signature, then the action of e_i (resp. f_i) should be set to 0. The value of $\varepsilon_i(b)$ (resp. $\varphi_i(b)$) is given by the number of –(resp. +) in the reduced signature.

Example 2.6 Since the signature rule enables us to calculate the multiple tensor product of $B^{n,l}$'s, we consider $B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. Let $c^{(j)}(j = 1, ..., 5)$ as in Example 2.4.

Consider an element $b = (c_0 \ c^{(1)}c^{(1)}c^{(2)}) \otimes (c_0 \ c_0 \ c^{(1)}) \otimes (c^{(1)}c^{(3)}) \in B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. The 4-signature is given as follows

 $b = (c^{(2)} \otimes c^{(1)} \otimes c^{(1)} \otimes c_0) \otimes (c^{(1)} \otimes c_0 \otimes c_0) \otimes (c^{(3)} \otimes c^{(1)})$ $\eta_4 = - - + - + - + - -$

 $2 \ 3 \ 6$

The reduced signature is $\eta_4 = - - +$, where the upper number signifies the component of the tensor product the sign belonged to. Therefore, we have

 $e_4b = (c^{(2)} \otimes c^{(1)} \otimes e_4(c^{(1)}) \otimes c_0) \otimes (c^{(1)} \otimes c_0 \otimes c_0) \otimes (c^{(3)} \otimes c^{(1)}) = (c_0 \ c_0 \ c^{(1)}c^{(2)}) \otimes (c_0 \ c_0 \ c^{(1)}) \otimes (c^{(1)}c^{(3)})$

$$\begin{aligned} f_4b &= (c^{(2)} \otimes c^{(1)} \otimes c^{(1)} \otimes c_0) \otimes (c^{(1)} \otimes f_4(c_0) \otimes c_0) \otimes (c^{(3)} \otimes c^{(1)}) = (c_0 \ c^{(1)} c^{(1)} c^{(2)}) \otimes (c_0 \ c^{(1)} c^{(1)}) \otimes (c^{(1)} c^{(3)}) \end{aligned}$$



Example 2.7 Since the signature rule enables us to calculate the multiple tensor product of $B^{n,l}$'s, we consider $B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. Let $\tilde{c}^{(j)}(j = 1, ..., 5)$ as in Example 2.5.

Consider an element $b' = (\tilde{c}_0 \ \tilde{c}^{(1)} \tilde{c}^{(2)}) \otimes (\tilde{c}_0 \ \tilde{c}_0 \ \tilde{c}^{(1)}) \otimes (\tilde{c}^{(1)} \tilde{c}^{(3)}) \in B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. The 5-signature is given as follows

 $b' = (\tilde{c}^{(2)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}_0) \otimes (\tilde{c}^{(1)} \otimes \tilde{c}_0 \otimes \tilde{c}_0) \otimes (\tilde{c}^{(3)} \otimes \tilde{c}^{(1)})$ $\eta_5 = - + + - + + -$

 $2 \ 3 \ 6$

The reduced signature is $\eta_5 = - - +$, where the upper number signifies the component of the tensor product the sign belonged to. Therefore, we have

 $e_{5}b' = (\tilde{c}^{(2)} \otimes \tilde{c}^{(1)} \otimes e_{4}(\tilde{c}^{(1)}) \otimes \tilde{c}_{0}) \otimes (\tilde{c}^{(1)} \otimes \tilde{c}_{0} \otimes \tilde{c}_{0}) \otimes (\tilde{c}^{(3)} \otimes \tilde{c}^{(1)}) = (\tilde{c}_{0} \ \tilde{c}_{0} \ \tilde{c}^{(1)} \tilde{c}^{(2)}) \otimes (\tilde{c}_{0} \ \tilde{c}_{0} \ \tilde{c}^{(1)}) \otimes (\tilde{c}^{(1)} \tilde{c}^{(3)})$

 $\begin{aligned} f_5b' &= (\tilde{c}^{(2)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}_0) \otimes (\tilde{c}^{(1)} \otimes f_4(\tilde{c}_0) \otimes \tilde{c}_0) \otimes (\tilde{c}^{(3)} \otimes \tilde{c}^{(1)}) = (\tilde{c}_0 \ \tilde{c}^{(1)} \tilde{c}^{(1)} \tilde{c}^{(2)}) \otimes \\ (\tilde{c}_0 \ \tilde{c}^{(1)} \tilde{c}^{(1)}) \otimes (\tilde{c}^{(1)} \tilde{c}^{(3)}) \end{aligned}$

2.4 $B^{n,l}$ and $B^{n-1,l}$ of type $D_n^{(1)}$

We give an affine crystal action on $B^{n,l}$. To do this we need $B^{n-1,l}$ of type $D_n^{(1)}$. $B^{n,l}$ and $B^{n-1,l}$ are associated to the spin nodes in the Dynkin diagram. As $\{1, 2, ..., n\}$ crystals we have the isomorphisms

$$B^{n,l} \cong B(l\Lambda_n), \quad B^{n-1,l} \cong B(l\Lambda_{n-1}).$$
 (2.6)

To define the affine crystal action, we introduce an involution $\sigma : B^{n,l} \leftrightarrow B^{n-1,l}$ corresponding to the Dynkin diagram automorphism that interchanges the nodes n to n-1. Let $J = \{2, 3, ..., n\}$. J is the J-highest if and only if $e_i b = 0$ for every $i \in J$. By definition in [2], σ is required to commute with e_i , $f_i (i \in J)$. Hence it sufficies to define σ on J-highest elements in $B^{n,l}$ and all of the form of the LHS of (2.7) with some a, and mapped by σ as

$$\sigma: \underbrace{\begin{bmatrix} n \cdots n & \bar{1} \cdots \bar{1} \\ \vdots & \bar{n} \cdots \bar{n} \\ 2 \cdots 2 & \vdots \\ 1 \cdots 1 & 2 \cdots 2 \\ a & l-a \end{array}}_{a & l-a} \longleftrightarrow \underbrace{\begin{bmatrix} \bar{n} \cdots \bar{n} & \bar{1} \cdots \bar{1} \\ \vdots & n \cdots n \\ 2 \cdots 2 & \vdots \\ 1 \cdots 1 & 2 \cdots 2 \\ l-a & a \end{bmatrix}}_{l-a} (2.7)$$



Example 2.8 When n = 4, the crystal graph of $B^{3,1}$ of type $D_4^{(1)}$ is depicted as follows.



Definition 2.9 The action e_0 and f_0 on $B^{n,l}$ is given by

$$e_0 = \sigma \circ e_1 \circ \sigma, \qquad f_0 = \sigma \circ f_1 \circ \sigma$$

Example 2.10 Consider the case $B^{n,l}$'s, where n = 4 and l = 5. Let $c^{(j)}$ (j = 1, ..., 5) as in Example 2.4.

Consider an element $b = (c_0 \ c_0 \ c^{(3)} c^{(3)}) \in B^{4,5}$. We are to calculate $e_0 b$.

$$\sigma(b) = \begin{bmatrix} \bar{4} \ \bar{4} \ \bar{1} \ \bar{1} \ \bar{1} \\ 3 \ 3 \ 4 \ 4 \ 4 \\ 2 \ 2 \ 3 \ 3 \ 3 \\ 1 \ 1 \ 2 \ 2 \ 2 \end{bmatrix}$$

$$e_1 \begin{bmatrix} \bar{4} \ \bar{4} \ \bar{1} \ \bar{1} \ \bar{1} \\ 3 \ 3 \ 4 \ 4 \ 4 \\ 2 \ 2 \ 3 \ 3 \ 3 \\ 1 \ 1 \ 2 \ 2 \ 2 \end{bmatrix} = \begin{bmatrix} \bar{4} \ \bar{4} \ \bar{2} \ \bar{1} \ \bar{1} \\ 3 \ 3 \ 4 \ 4 \ 4 \\ 2 \ 2 \ 3 \ 3 \ 3 \\ 1 \ 1 \ 1 \ 2 \ 2 \end{bmatrix}$$

$$(2.8)$$

$$f_{1} \begin{bmatrix} \bar{4} & \bar{4} & \bar{1} & \bar{1} & \bar{1} \\ 3 & 3 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix} = 0$$
(2.10)

But, σ is required to commute with $e_i, f_i (i \in J)$. Then we apply e_3e_2 to get the J-highest.

$$e_{3}e_{2} \begin{bmatrix} \bar{4} \ \bar{4} \ \bar{2} \ \bar{1} \ \bar{1} \\ 3 \ 3 \ 4 \ 4 \ 4 \\ 2 \ 2 \ 3 \ 3 \ 3 \\ 1 \ 1 \ 2 \ 2 \end{bmatrix} = \begin{bmatrix} \bar{4} \ \bar{4} \ \bar{4} \ \bar{1} \ \bar{1} \\ 3 \ 3 \ 3 \ 4 \ 4 \\ 2 \ 2 \ 3 \ 3 \\ 1 \ 1 \ 1 \ 2 \ 2 \end{bmatrix} = b'.$$

$$(2.11)$$

$$Since \ \sigma(f_{2}f_{3}(b')) = f_{2}f_{3}\sigma(b') \ and \ \sigma(b') = \begin{bmatrix} 4 \ 4 \ \bar{1} \ \bar{1} \ \bar{1} \\ 3 \ 3 \ \bar{4} \ \bar{4} \\ 2 \ 2 \ 3 \ 3 \\ 1 \ 1 \ 2 \ 2 \ 2 \end{bmatrix} = b'.$$

$$e_{0}b = f_{2}f_{3}\sigma(b') = f_{2}f_{3} \begin{bmatrix} 4 \ 4 \ \bar{1} \ \bar{1} \ \bar{1} \\ 3 \ 3 \ \bar{4} \ \bar{4} \ \bar{4} \\ 2 \ 2 \ 3 \ 3 \ 3 \\ 1 \ 1 \ 2 \ 2 \ 2 \end{bmatrix} = \begin{bmatrix} 4 \ 4 \ \bar{1} \ \bar{1} \ \bar{1} \\ 3 \ 3 \ \bar{4} \ \bar{4} \ \bar{4} \\ 2 \ 2 \ 3 \ 3 \ 3 \\ 1 \ 1 \ 2 \ 2 \ 2 \end{bmatrix}$$

$$f_{0}b = 0.$$

2.5 Energy function

Next consider a \mathbb{Z} -valued function H on $B \otimes B'$ satisfying the following property: For any $b \in B, b' \in B'$ and i such that $e_i(b \otimes b') \neq 0$

$$H(e_i(b \otimes b')) = \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0 \ , \ \varphi_0(b) \ge \varepsilon_0(b'), \ \varphi_0(\tilde{b'}) \ge \varepsilon_0(\tilde{b}), \\ H(b \otimes b') - 1 & \text{if } i = 0 \ , \ \varphi_0(b) < \varepsilon_0(b'), \ \varphi_0(\tilde{b'}) < \varepsilon_0(\tilde{b}), \\ H(b \otimes b') & \text{otherwise.} \end{cases}$$

H is known to exist and unique up to additive constant. \tilde{b} and \tilde{b}' are defined from the combinatorial *R* matrix by $R(b \otimes b') = \tilde{b}' \otimes \tilde{b}$. The existences of the isomorphism and energy function *H* are guaranteed by the existence of the *R* matrix. See Ref [13].

Note that we normalized H so that we have $H((c_0)^l \otimes (c_0)^{l'}) = 0$. Here and later, $(c_0)^l$ means $\underbrace{c_0 \cdots c_0}_{l \to \infty} \in B^{n,l}$.

Definition 2.11 A combinatorial R matrix for the crystal $B^{n,s} \otimes B^{n,t}$ is a map

$$R: B^{n,s} \otimes B^{n,t} \to B^{n,t} \otimes B^{n,s}$$

satisfying

(1) $R \circ e_i = e_i \circ R$, $R \circ f_i = f_i \circ R$ for i = 1, ..., n, and

(2)



 $n' = \left[\frac{n}{2}\right]$ and $t_1 + t_2 + \dots + t_{n'} \le s$, $s' = s - (t_1 + t_2 + \dots + t_{n'})$.

We explain how to calculate R(b) for a general element b. Let $b \xrightarrow{e_{a_1}} b_1 \xrightarrow{e_{a_2}} b_2 \cdots b_{m-2} \xrightarrow{e_{a_{m-1}}} b_{m-1} \xrightarrow{e_{a_m}} \tilde{b}$, where $e_i(\tilde{b}) = 0$ for every $i \neq 0$. Namely, \tilde{b} is of the form



$$R(b) = R(f_{a_1} \cdots f_{a_m} \tilde{b}) = f_{a_1} \cdots f_{a_m} R(\tilde{b}).$$

Example 2.12

$$\begin{aligned} &Set \ b = \begin{bmatrix} \overline{3} & \overline{3} & \overline{1} \\ \overline{4} & \overline{4} & \overline{3} \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \otimes \begin{bmatrix} \overline{2} & \overline{2} \\ \overline{4} & \overline{3} \\ 3 & 4 \\ 1 & 1 \end{bmatrix} and \ we \ calculate \ R(b). \\ &\tilde{b} = e_4 e_2^3 e_4^2 e_3^2 e_1 b = \begin{bmatrix} 4 & 4 & 4 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} \overline{3} & \overline{3} \\ \overline{4} & \overline{4} \\ 2 & 2 \\ 1 & 1 \end{bmatrix} and \ R(\tilde{b}) = \begin{bmatrix} 4 & 4 \\ 3 & 3 \\ 2 & 2 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 4 & 4 \\ 3 & 3 \\ 2 & 2 \\ 1 & 1 \end{bmatrix} \\ Since \ f_1 f_3^2 f_4^2 f_2^3 f_4 R(\tilde{b}) = \begin{bmatrix} \overline{3} & \overline{3} \\ \overline{4} & \overline{4} \\ 2 & 2 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} \overline{2} & \overline{2} & \overline{1} \\ \overline{4} & \overline{3} & \overline{3} \\ 3 & 4 & 4 \\ 1 & 1 & 2 \end{bmatrix} \\ we \ have \ R\left(\begin{bmatrix} \overline{3} & \overline{3} & \overline{1} \\ \overline{4} & \overline{4} & \overline{3} \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \otimes \begin{bmatrix} \overline{2} & \overline{2} & \overline{1} \\ \overline{4} & \overline{3} & \overline{3} \\ 3 & 4 & 4 \\ 1 & 1 & 2 \end{bmatrix} \right) = \begin{bmatrix} \overline{3} & \overline{3} \\ \overline{4} & \overline{4} \\ 2 & 2 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} \overline{2} & \overline{2} & \overline{1} \\ \overline{4} & \overline{3} & \overline{3} \\ 3 & 4 & 4 \\ 1 & 1 & 2 \end{bmatrix} . \end{aligned}$$

Lemma 2.13 We give all the images of the isomorphism
$$B^{n,l} \otimes B^{n,1} \simeq B^{n,1} \otimes B^{n,l}$$

that will be needed in the next section. Let the symbol

$$\alpha + \alpha' + \alpha' \\ \beta'$$

signify $\alpha \otimes \beta \mapsto \beta' \otimes \alpha'$ under the isomorphism. We also set

	$\overline{n-1}$			n-2			$\overline{n-3}$	-		$\overline{n-3}$			$\overline{n-3}$;
	\bar{n}			\bar{n}			$\overline{n-2}$			$\overline{n-1}$	-		$\overline{n-2}$,
	n-2			n-1			n			n			$\overline{n-1}$	
$c_a =$			$c_h =$			$c_d =$			$c_e =$	•		$c_f =$	\bar{n}	
-u	•	,	-0	•	,	u	•	,	~6	•	,	٥J	•	
				•			•							
	2			2			2			2				
	1			1			1			1			1	

and c_0 is the column of height n without barred letters.

We list up the cases as below where $i, j, k, l, m, p \in \mathbb{Z}_{\geq 0}$. Case 1. (i) m > 0(ii) l > 0 $c_0^i c_a^j c_b^k c_d^l c_f^m \underset{c_f}{\overset{c_0}{+}} c_0^{i+1} c_a^j c_b^k c_d^l c_f^{m-1}$ c_0^{i+1} c_d (iv) j > 0 c_0 (v) i > 0(iii) k > 0 $c_0^i c_a^j c_b^k \stackrel{c_0}{\underset{c_b}{+}}$ $\begin{array}{c}c_0\\c_0^i+c_0^i\end{array}$ c_0 Case 2. (i) i > 0, m > 0(ii) m > 0 $c_a + c_a$ $c_a^j c_b^k c_d^l c_f^m \stackrel{c_a}{\underset{c_f}{+}} c_a^{j+1} c_b^k c_d^l c_f^{m-1}$ $+1c_{b}^{k}c_{d}^{l+1}c_{f}^{m-1}$ $c_0^i c_a^j c_b^k c_d^l c_f^m$ (iv) l > 0(iii) i > 0 $c_0^i c_a^j c_b^k c_d^l \stackrel{c_a}{\underset{c_0}{+}} c_0^{i-1} c_a^{j+1} c_b^k c_d^l$ $c_a^j c_b^k c_d^l \stackrel{c_a}{\underset{c_d}{+}} c_a^{j+1} c_b^k c_d^{l-1}$ (v) k > 0(vi) j > 0> c $c_a^j \stackrel{c_a}{\underset{c_a}{\overset{c_a}{\leftarrow}}} c_a^j$ $c_a^j c_b^k \stackrel{c_a}{\underset{c_b}{+}} c_a^{j+1} c_b^{k-1}$

Case 3.



$$c_{0}^{i}c_{d}^{l}c_{f}^{m} \stackrel{c_{d}}{\underset{c_{a}}{+}} c_{0}^{i-1}c_{d}^{l+2}c_{f}^{m-1} \qquad c_{0}^{i}c_{a}^{j}c_{d}^{l} \stackrel{c_{d}}{\underset{c_{d}}{+}} c_{0}^{i+1}c_{a}^{j-1}c_{d}^{l-1}c_{f} \qquad c_{0}^{i}c_{d}^{l} \stackrel{c_{d}}{\underset{c_{0}}{+}} c_{0}^{i-1}c_{d}^{l+1}c_{f} \qquad c_{0}^{i}c_{d}^{l} \stackrel{c_{d}}{\underset{c_{0}}{+}} c_{0}^{i-1}c_{d}^{l+1}c_{f} \qquad c_{0}^{i}c_{d}^{l} \stackrel{c_{0}}{\underset{c_{0}}{+}} c_{0}^{i-1}c_{d}^{l+1}c_{f} \qquad c_{0}^{i-1}c_{d}^{l+1}c_{f} \qquad c_{0}^{i-1}c_{d}^{l+1}c_{f} \qquad c_{0}^{i-1}c_{d}^{i-1}c_{f} \qquad c_{0}^{i-1}c_{f} \qquad c_{0}^{i-1}c_{i$$

$$\begin{array}{ll} ({\rm x}) \ j>0, \ l>0 & ({\rm xi}) \ i>0, \ m>0 & ({\rm xii}) \ j>0 \\ c_a^j c_d^l c_f^m {\stackrel{c_d}{+}} \ c_0^i c_a^{j-1} c_d^{l-1} c_f^{m+1} & c_0^i c_f^m {\stackrel{c_d}{+}} \ c_0^{i-1} c_d^2 c_f^{m-1} & c_0^i c_a c_f^m {\stackrel{c_d}{+}} \ c_0^i c_d c_f^m \\ \end{array}$$

$$\begin{array}{ll} \text{(xiii)} \ i > 0, \ m > 0 & \text{(ivx)} \ m > 0 & \text{(ix)} \ l > 0 \\ c_0^i c_d^l c_f^m \underset{c_a}{\overset{c_d}{\leftarrow}} c_0^{i-1} c_d^{l+2} c_f^{m-1} & c_d^l c_f^m \underset{c_f}{\overset{c_d}{\leftarrow}} c_d^{l+1} c_f^{m-1} & c_d^l \underset{c_d}{\overset{c_d}{\leftarrow}} c_d^l \\ \end{array}$$

Case 5.
(i)
$$i > 0$$

 $c_0^i c_a^j c_b^k c_d^l c_f^m \stackrel{c_f}{+} c_0^{i-1} c_a^j c_b^k c_d^l c_f^{m+1}$
(ii) $k > 0$
 $c_a^j c_b^k c_d^m \stackrel{c_f}{+} c_d^j c_b^k c_d^{l-1} c_f^{m+1}$
(iv) $j > 0$
 $c_a^j c_b^k c_f^m \stackrel{c_f}{+} c_d^j c_b^{k-1} c_f^{m+1}$
 $c_a^j c_b^k c_f^m \stackrel{c_f}{+} c_d^j c_b^{k-1} c_f^{m+1}$
(iv) $j > 0$
 $c_a^j c_b^k c_f^m \stackrel{c_f}{+} c_d^j c_b^{k-1} c_f^{m+1}$
(iv) $j > 0$
(v) $m > 0$
 $c_a^j c_b^m \stackrel{c_f}{+} c_a^j c_f^{m+1}$
(v) $c_a^j c_f^m \stackrel{c_f}{+} c_f^j c_f^m$



 c_b

Proof for Case 1 where k > 0. $(c_b^k \otimes c_a^j \otimes c_0^i) \otimes c_0 \xrightarrow{e_{n-2}^k} (c_a^{j+k} \otimes c_0^i) \otimes c_0 \xrightarrow{e_n^{j+k}} c_0^{i+j+k} \otimes c_0$. Reform $c_0^{i+j+k} \otimes c_0$ to $c_0 \otimes (c_0 \otimes c_0^{i+j+k})$. Then $c_0 \otimes c_0^{i+j+k} \xrightarrow{f_n^{j+k}} c_a \otimes (c_a^{j+k-1} \otimes c_0^{i+1}) \xrightarrow{f_{n-2}^k} c_b \otimes (c_b^{k-1} \otimes c_a^j \otimes c_0^{i+1})$. Hence $R(c_0^i c_a^j c_b^k \otimes c_0) = c_b \otimes c_0^{i+1} c_a^j c_b^{k-1}$.

 c_a

Definition 2.14 A combinatorial R matrix for the crystal $B \otimes B'$ is a map R: $Aff(B) \otimes Aff(B') \rightarrow Aff(B') \otimes Aff(B)$ given by

$$R(z^{d}b \otimes z^{d'}b') = z^{d'+H(b \otimes b')}\tilde{b'} \otimes z^{d-H(b \otimes b')}\tilde{b}$$

where $b \otimes b' \mapsto \tilde{b}' \otimes \tilde{b}$ under the isomorphism $B \otimes B' \xrightarrow{\sim} B' \otimes B$. The following result is a direct consequence of the ordinary (i.e. not combinatorial) Yang-Baxter equation.

2.6 Yang-Baxter equation

Let us define the affinization Aff(B) of the crystal B. We introduce an indeterminate z (the spectral parameter) and set

$$\operatorname{Aff}(B) = \{ z^d b \mid d \in \mathbb{Z}, b \in B \}.$$

Thus Aff(B) is an infinite set. $z^0 b \in Aff(B)$ will often be written as b. Aff(B) also admits the crystal structure by $e_i \cdot z^d b = z^{d+\delta_{i0}}(e_i b), f_i \cdot z^d b = z^{d-\delta_{i0}}(f_i b).$

Proposition 2.15 (Yang-Baxter equation). Let $B^{n,l} = B_l$. The following equation hold on $Aff(B_l) \otimes Aff(B_{l'}) \otimes Aff(B_{l''})$.

$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R)$$

3 Soliton cellular automata

3.1 States and time evolutions.

Consider the crystal $(B^{n,1})^{\otimes N}$ for sufficiently large N. The elements of $(B^{n,1})^{\otimes N}$ we have in mind are of the following form:

$$\cdots \otimes c_0 \otimes \cdots \otimes c_0 \otimes c_1 \otimes \cdots \otimes c_l \otimes c_0 \otimes \cdots \otimes c_0 \otimes \cdots$$

Namely, relatively few elements are non c_0 , and almost all are c_0 . In the assertions below, we embed, if necessary, $(B^{n,1})^{\otimes N}$ into $(B^{n,1})^{\otimes N'}(N < N')$ by

 $(B^{n,1})^{\otimes N} \hookrightarrow (B^{n,1})^{\otimes N'},$

$$b_1 \otimes \cdots \otimes b_N \mapsto b_1 \otimes \cdots \otimes b_N \otimes \underbrace{c_0 \otimes \cdots \otimes c_0}_{N'-N}.$$

Lemma 3.1 By iterating $B^{n,l} \otimes B^{n,1} \to B^{n,1} \otimes B^{n,l}$ we consider a map

 $B^{n,l} \otimes B^{n,1} \otimes \cdots \otimes B^{n,1} \xrightarrow{\sim} B^{n,1} \otimes \cdots \otimes B^{n,1} \otimes B^{n,l},$ $(c_0)^l \otimes b_1 \otimes \cdots \otimes b_N \mapsto \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_N \otimes \tilde{b},$

then there exists an integer N_0 such that $\tilde{b} = (c_0)^l$ for $N \ge N_0$.

Taking sufficiently large N such that the above lemma holds, we define a map $T_l: (B^{n,1})^{\otimes N} \longrightarrow (B^{n,1})^{\otimes N}$ by $b_1 \otimes \cdots \otimes b_N \mapsto \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_N$.

Lemma 3.2 For a fixed element of $(B^{n,1})^{\otimes N}$ as a Lemma 3.1, there exists an integer l_0 such that $T_l = T_{l_0}$ for any $l \geq l_0$.

Both lemmas are obvious from Lemma 2.13.

An element of $(B^{n,1})^{\otimes N}$ having the property described in the beginning of this subsection will be called a *state*. Lemma 3.1 and Lemma 3.2 enable us to define an operator $T = \lim_{l\to\infty} T_l$ on the space of states. Application of T induces a transition of state. Thus it can be regarded as a certain dynamical system, in which T plays the role of 'time evolution'. By the same reason, T_l may also be viewed as another time evolution. (In this paper, time evolution means the one by T unless otherwise stated.)

3.2 Conservation laws

Fix sufficiently large N and consider a composition of the combinatorial R matrices

$$R_l = R_{NN+1} \cdots R_{23} R_{12} : \operatorname{Aff}(B^{n,l}) \otimes \operatorname{Aff}(B^{n,1})^{\otimes N} \to \operatorname{Aff}(B^{n,1})^{\otimes N} \otimes \operatorname{Aff}(B^{n,l})$$

Here R_{ii+1} signifies that the R matrix acts on the *i*-th and (i + 1)-th components of the tensor product. Applying R_l to an element $(c_0)^l \otimes p$ $(p = b_1 \otimes \cdots \otimes b_N)$, we have

$$R_l((c_0)^l \otimes p) = z^{H_1} \tilde{b}_1 \otimes z^{H_2} \tilde{b}_2 \otimes \cdots \otimes z^{H_N} \tilde{b}_N \otimes z^{\tilde{E}_l(p)}(c_0)^l,$$
$$\tilde{E}_l(p) = -\sum_{j=1}^N H_j, \quad H_j = H(b^{(j-1)} \otimes b_j),$$

where $b^{(0)} = (c_0)^l$ and $b^{(j)}$ $(1 \le j < N)$ is defined by

$$B^{n,l} \otimes \underbrace{B^{n,1} \otimes \cdots \otimes B^{n,1}}_{j} \simeq \underbrace{B^{n,1} \otimes \cdots \otimes B^{n,1}}_{j} \otimes B^{n,l},$$
$$(c_0)^l \otimes b_1 \otimes \cdots \otimes b_i \mapsto \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_i \otimes b^{(j)}.$$

Lemma 3.3 Let H be the energy function and $-H_j = -H(b^{(j-1)} \otimes b_j) \in \{0, 1, 2, ..., n'\}$. Then $b^{(j-1)} \otimes b_j$ commute with e_i , f_i until it will get in form $c_0^i c_d^l \otimes c_d \mapsto c_0 \otimes c_0^{i-1} c_d^{l+1}$. as in Case 4 in Lemma 2.13. Then by using the rule in [24] we will get $-H_j = 1$ if and only if $c_0^i c_d^l \otimes c_d \mapsto c_0 \otimes c_0^{i-1} c_d^{l+1}$.

Proof. Use the definition of energy function in [31] to prove it directly.



Proposition 3.4 For an element $p \in (B^{n,1})^{\otimes N}$, we have (1) $T_l T_{l'}(p) = T_{l'} T_l(p)$. (2) $E_l(T_{l'}(p)) = E_l(p)$. In particular, $E_l(T(p)) = E_l(p)$.

We refer to [3] for the proof.

3.3 Soliton

A state of the following form is called an m soliton state of length $l_1, l_2, ..., l_m$,

$$\dots [l_1]\dots\dots [l_2]\dots\dots\dots [l_m]\dots\dots$$
(3.1)

Here $\cdots [l] \cdots$ denotes a local configuration such as

$$\cdots \otimes c_0 \otimes c_0 \otimes \underbrace{c_l \otimes c_{l-1} \otimes \cdots \otimes c_1}_{l} \otimes c_0 \otimes c_0 \otimes \cdots, \qquad (c_0 \preceq c_1 \preceq c_2 \preceq \cdots \preceq c_l)$$

where c_0 means column without barred letter and $c_j \in B^{n,1}$ with exactly two barred letters. That means we cannot form the soliton as above when the number of barred letter is more than two. $c \leq c'$ if and only if a_j (*j*-th entry of *c*), $a_j \leq a'_j$ for all j = 1, 2, ..., n.

Remark. It would be an interesting problem to consider color separation scheme in [[34]: §4.7]. A reasonable choice of B_{\natural} such as $B^{n-1,1}$ or $B^{n-2,1}$ seems to fail for n = 4.

Lemma 3.5 Let p be a one-soliton state of length l, then

- (1) The kth conserved quantity of p is given by $E_k(p) = min(k, l)$.
- (2) The state $T_k(p)$ is obtained by the rightward shift by $E_k(p)$ lattice steps.

Proof. (1) Recall that the conserved quantity E_k is a sum of local H functions $-H_j = -H(b^{(j-1)} \otimes b_j)$. $b^{(j-1)} \otimes b_j \mapsto \tilde{b}_j \otimes b^{(j)}$ commutes with $f_i (i \in 1, 2, ..., n-1)$ until we will get the lowest weight where $f_i b = 0$ for any $i \in \{1, 2, ..., n-1\}$. Then we apply f_0, e_0 and we will get $(c_0)^k \otimes c_0 \mapsto c_0 \otimes (c_0)^k$. By using Lemma 3.3, $-H_j = 1$ if and only if $(c_0)^i c_d^l \otimes c_d \mapsto c_0 \otimes (c_0)^{i-1} c_d^{l+1}$. $E_k(p)$ is the sum of the local $-H_j = 1$ in the tensor product in $B^{n,l}$. Hence $E_k = \min(k, l)$. Similarly, the statement (2) follows from the rule (2.3) and Lemma 3.1.

Definition 3.6 For any state p, the number $N_l = N_l(p)$ (l = 1, 2, ...) are defined by

$$E_l = \sum_{k>1} \min(k, l) N_k, \qquad E_0 = 0, \qquad N_i = -E_{l-1} + 2E_l - E_{l+1}.$$



By Lemma 3.5, we have

Proposition 3.7 For m-soliton state (3.1), N_l is the number of solitons of length l,

$$N_l = \#\{j|l_j = l\}.$$

This proposition implies the stability of solitons, since the number $E_l(p)$, and hence $N_l(p)$, are conserved.

3.4 Type A $B^{r,s}$

In this subsection, we recall the crystal structure of $B^{r,s}$ for arbitrary r, s and the combinatorial R for $B^{r,s} \otimes B^{r',s'}$. Our reference is [31]. We use the French notation for semistandard tableau, which is upside-down from [31]

Our $U'_q(A^{(1)}_{n-1})$ -crystal $B^{r,s}$ $(1 \le r \le n-1, s \in \mathbb{Z}_{>0})$ is as a set, identified with the set of semistandard tableau of rectangular shape (s^r) with letters from $\{1, 2, ..., n\}$. For an element t of $B^{r,s}$, let t_{ij} denote the letter in the *i*-th row from bottom and *j*-th column of t. We first describe the action of e_i , f_i for i = 1, 2, ..., n-1. For this purpose, let us define the **Japanese reading word** of t by

$$J(t) = w^{(1)}w^{(2)}\cdots w^{(s)}, \quad w^{(j)} = t_{1j}t_{2j}\cdots t_{rj} \quad (j = 1, 2, ..., s).$$

We then regard J(t) as an element of $(B^{1,1})^{\otimes (rs)}$. Namely, each letter is considered to be an element of $B^{1,1}$.

The bumping algorithm is defined for a pair of tableau t and single word u and depicted as $t \leftarrow u$. First, let us consider the case where t is one-row tableau. If t is empty, $t \leftarrow u$ is defined to be the tableau u with one node. Otherwise, let $t = t_{11}t_{12}\cdots t_{1m}$ and look at

$$t_{11}t_{12}\cdots t_{1m} \leftarrow u.$$

If $t_{1m} \leq u$, then define

$$t \leftarrow u = t_{11}t_{12}\cdots t_{1m}u$$

and the algorithm stops (case (a)). Otherwise, set $i_1 = \min\{i \mid t_{1i} > u\}$ and define



$$t \leftarrow u = t_{11} \cdots t_{1i_1 - 1} u t_{1i_1 + 1} \cdots t_{1m}$$

and we have the single word t_{1i_1} bumped out from t (case (b)). Now suppose we have a tableau t of l rows and let t_i be the *i*-th row of t. The bumping algorithm $t \leftarrow u$ proceeds as follows. Set $t'_1 = t_1 \leftarrow u$. If case (a) occurs, the algorithm stops. Otherwise, let u_1 be the letter bumped out and set $t'_2 = t_2 \leftarrow u_1$. We again devide the algorithm into the two cases. The algorithm proceeds until it stops. If case (b) still occurs in the highest row, we append the empty row above it.

Example 3.8 Let

$$t = \frac{23444}{11122}$$

and u = 1. The bumping algorithm proceeds as follows. TUNKU TUN AMINA

23444		2 3 444	\leftarrow	2
111 2 2	$\leftarrow 1$	11112		

And we have the answer.

22444 11112

3

Example 3.9 Let

$$t = \frac{24555}{11334}$$

and u = 2. The bumping algorithm proceeds as follows.

24555		24555	\leftarrow	3
11 <i>3</i> 34	$\leftarrow 2$	11234		

And we have the answer.

For a tableau $t \in B^{r,s}$ we define the **row word** row t by

$$row(t) = t_r t_{r-1} \cdots t_1$$
 $t_i = t_{i1} t_{i2} \cdots t_{is}$ $(i = 1, 2, ..., r).$

Let t be a tableau and $w = u_1 u_2 \cdots u_l$ a word of length l. Let $t \leftarrow w$ be a tableau obtained by applying the bumping algorithm for a single word u_j successively as

$$(\cdots ((t \leftarrow u_1) \leftarrow u_2) \leftarrow \cdots) \leftarrow u_l$$

Then we have the following proposition to obtain the combinatorial R for $B^{r,s} \otimes B^{r,m}$.

Proposition 3.10 [31] Assume $t \in B^{r,m}$ and $t' \in B^{r,s}$. Then $t' \otimes t$ is mapped to $\tilde{t} \otimes \tilde{t}'$ by the crystal isomorphism

$$B^{r,s} \otimes B^{r,m} \longrightarrow B^{r,m} \otimes B^{r,s}$$
 if and only if $t \leftarrow row(t') = \tilde{t}' \leftarrow row(\tilde{t})$.

Moreover, the energy function $H(t' \otimes t)$ is given by the number of nodes in the shape of $t \leftarrow row(t')$ that are strictly north of the r-th row.

Note that the decomposition of $B^{r,s} \otimes B^{r,m}$ into $U_q(A_{n-1}^{(1)})$ -crystals is multiplicity free. From this fact, it follows that for a given pair $t' \otimes t$ we can determine \tilde{t}, \tilde{t}' uniquely. To explain the algorithm of computing \tilde{t}, \tilde{t}' we prepare terminology. Let θ be a skew tableau, that is, set-theoretical difference of a Young diagram from a smaller one with letters in each node. Let τ be the shape of θ . θ is called a vertical k-strip if $|\tau| = k$ and $\tau_i \leq 1$ for any $i \geq 1$. The algorithm to obtain \tilde{t}, \tilde{t}' is given as follows. Let p be the tableau obtained by the bumping algorithm $t \leftarrow row(t')$. We attach an integer from 1 to rm to each node of the skew tableau p - p', where p' is the NE part of p whose shape is (s^r) . The integer should be labeled in the following manner. Let θ_1 be the rightmost vertical r-strip in p - p' as lower as possible. We attach integers 1 through r from lower nodes. Remove θ_1 from p - p' and define the vertical r-strip θ_2 in similar manner. Continue it until we finish attaching all integers up to rm. Next we apply the reverse bumping algorithm according to the order of the labeling. Namely we find a word u_1 and a tableau p_1 whose shape is (shape of p)-(node of label 1), such that $p_1 \leftarrow u_1 = p$. (Note that such a pair p_1, u_1) is unique.) We repeat this procedure to obtain u_2 and p_2 by replacing p and the node of label 1 with p_1 and the node of label 2 and continue until we arrive at a tableau of shape (s^r) . Then we have

 $\tilde{t} = ((\cdots (\phi \leftarrow u_{rm}) \leftarrow \cdots) \leftarrow u_2) \leftarrow u_1 \text{ and } \tilde{t}' = p_{rm}.$

Note that in [31], the energy function $H(t' \otimes t)$ is given by the number of nodes in the shape of $t' \leftarrow \operatorname{row}(t)$ that are strictly east of the $\max(m, s)$ -th column.

We introduce ν as a map sending an element b of the $U'_q(D_n^{(1)})$ -crystal $B^{n,l}$ to νb in the $U'_q(A_{n-1}^{(1)})$ -crystal $B_A^{2,l}$. The operator ν will change $D_n^{(1)}$ to $A_{n-1}^{(1)}$ as a set,



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List of Publications

List of Peer-Reviewed Publications

[1] Mahathir bin Mohamad, Scattering Rules in Soliton Cellular Automata Associated with $U_q(D_n^{(1)})$ -Crystal $B^{n,1}$, Journal of Physics A: Mathematical and Theoretical 45, (2012) 22 pages.

[2] K.C. Misra, M. Mohamad, M. Okado, Zero Action on Perfect Crystals for $U_q(G_2^{(1)})$, Symmetry, Integrability and Geometry: Method and applications 6, (2010) 022, 12 pages.

Seminar/ Workshop

[1] "Infinite Analysis 11 - Frontier of Integrability -", Poster Presentation, 2011.



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