Soliton Cellular Automata constructed from a $U_{q}\left(D_{n}^{(1)}\right)$-Crystal $B^{n, 1}$ and Kirillov-Reshetikhin type bijection for $U_{q}\left(E_{6}^{(1)}\right)$-Crystal $B^{6,1}$

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#### Abstract

In part 1 we study a class of cellular automata associated with the Kirillov-Reshetikhin crystal $B^{n, 1}$ of type $D_{n}^{(1)}$. They have a commuting family of time evolutions and solitons of length $l$ are labeled by $U_{q}\left(A_{n-1}^{(1)}\right)$-crystal $B_{A}^{2, l}$. The scattering rule of two solitons of lengths $l_{1}$ and $l_{2}\left(l_{1}>l_{2}\right)$ including the phase shift is identified with the combinatorial $R$-matrix for the $U_{q}\left(A_{n-1}^{(1)}\right)$-crystal $B_{A}^{2, l_{2}} \otimes B_{A}^{2, l_{1}}$. In part 2 we consider the Kirrilov-Reshetikhin crystal $B^{6,1}$ for the exceptional affine type $E_{6}^{(1)}$. We will give a conjecture on a statistic-preserving bijection between the highest weight paths consisting of $B^{6,1}$ and the corresponding rigged configuration. The algorithm only uses the structure of the crystal graph, hence could also be applied for other exceptional types. Our $B^{6,1}$ has a different algorithm compared our $B^{1,1}$ because we must consider the element $\phi$, unique element in the highest weight crystal of weight 0 , in the crystal graph. We will give many examples supporting the conjecture.


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Part I

## Scattering Rules in Soliton Cellular

 Automata Associated with $U_{q}\left(D_{n}^{(1)}\right)$-Crystal $B^{n, 1}$
## 1 Introduction

The box-ball system [33, 32] well-known as soliton cellular automata is a dynamical system of balls in a one dimensional array of boxes. The discrete KdV equation through a limiting procedure called ultradiscretization [36] was used to show the solitonic character like the KdV solitons. The rules for soliton interactions and factorization property of scattering matrices (Yang-Baxter equation) are justified by means of inverse ultradiscretization [35]. In [35] it is shown that the dynamical systems of soliton cellular automaton is described by an ultra-discrete equation obtained from extended Toda molecule equation. Later it was studied by [3] that the scattering of two solitons including the phase shift is described by isomorphism from the tensor product of two affine crystals for the quantum enveloping algebra $U_{q}\left(A_{n-1}^{(1)}\right)$ to the other order of the tensor product. The object they used is called combinatorial $R$-matrix [13]. The combinatorial $R$-matrix has an amazing property: it satisfies the Yang-Baxter equation, which assures that the scattering of three solitions does not depend on the order of scattering of the two solitons.

The new soliton cellular automata were constructed in [8] corresponding to $U_{q}\left(\mathfrak{g}_{n}\right)$ where $\mathfrak{g}_{n}=A_{2 n-1}^{(2)}, A_{2 n}^{(2)}, B_{n}^{(1)}, C_{n}^{(1)}, D_{n}^{(1)}, D_{n+1}^{(2)}$ and their internal degree of freedom was labeled by crystals of the smaller algebra $U_{q}\left(\mathfrak{g}_{n-1}\right)$. Then [7] studied the scattering rule of two solitons when they collide each other. They found that the scattering rule for affine crystals corresponding to $U_{q}\left(\mathfrak{g}_{n}\right)$ can be described by combinatorial $R$ matrix of the smaller algebra $U_{q}\left(\mathfrak{g}_{n-1}\right)$. The affine crystal they used is called KirillovReshetikhin (KR) crystal denoted by $B^{1,1}$. (The KR crystal is parameterized by two integers. The first index corresponds to a node of the Dynkin diagram of the affine algebra except 0 and the second a positive integer.) A generalization to the KR crystal $B^{k, 1}$ for $\mathfrak{g}=A_{n-1}^{(1)}$ was studied in [37] and their internal degree of freedom is given by the product of $U_{q}\left(A_{k-1}^{(1)}\right)$-crystal $B^{k-1, l}$ and $U_{q}\left(A_{n-k-1}^{(1)}\right)$-crystal $B^{1, l}$. A case for the exceptional algebra $\mathfrak{g}=D_{4}^{(3)}$ was also treated in [38]. In[38], it is shown that the scattering rule for the crystal type $U_{q}\left(D_{4}^{(3)}\right)$ is identified with the combinatorial $R$-matrix for $U_{q}\left(A_{1}^{(1)}\right)$-crystals and phase shifts are given by 3 -times of those in the well-known box ball system.

These results can be summarized and one might find the following conjectural properties for the solitons and their scatterings of the soliton cellular automaton constructed from the KR crystal $B^{k, 1}$ of the quantum affine algebra $U_{q}(\mathfrak{g})$. Let $G$ be the Dynkin diagram of the corresponding finite-dimensional simple Lie algebra of $\mathfrak{g}$. Let $\check{G}$ be the Dynkin diagram obtained by removing the node $k$ from $G$ and let $j$ be the node of $\check{G}$ that is connected to $k$ in $G$. Let $\mathfrak{g}$ be the corresponding affine algebra.

- The internal degree of freedom of soliton of length $l$ is described by the $U_{q}(\check{\mathfrak{g}})$ crystal $B^{j, l}$.
- The exchange of the internal degree of freedom by the scattering of solitons of length $l_{1}$ and $l_{2}\left(l_{1}>l_{2}\right)$ is given by the crystal isomorphism $B^{j, l_{1}} \otimes B^{j, l_{2}} \underset{\rightarrow}{\rightarrow} B^{j, l_{2}} \otimes B^{j, l_{1}}$.
- The phase shift of the scattering is described by the corresponding $H$ function.

This property allows us to calculate the combinatorial R matrix for $B^{j, l_{1}} \otimes B^{j, l_{2}}$ just by observing the scattering of solitons in the corresponding cellular automaton. If $\check{G}$ is not connected, then one needs to consider the tensor product as seen in [37]. Although this conjecture seems reasonable, the rigorous proof is yet to be given.

The purpose of this paper is to add another affirmative example to this conjecture. We take $\mathfrak{g}=D_{n}^{(1)}, k=n$. The corresponding node is a spin node and the KR crystal is $B^{n, 1}$. According to the above conjecture, we have $\mathfrak{g}=A_{n-1}^{(1)}, j=n-2$. In the crystal theory, there is a notion of the dual crystal. The dual $B^{\vee}$ of a crystal $B$ is defined by setting $\left(e_{i} b\right)^{\vee}=f_{i} b^{\vee},\left(f_{i} b\right)^{\vee}=e_{i} b^{\vee}$. Since we know $\left(B_{1} \otimes B_{2}\right)^{\vee}=B_{2}^{\vee} \otimes B_{1}^{\vee}$ and $\left(B^{j, l}\right)^{\vee}=B^{n-j, l}$ for the KR crystal of type $A_{n-1}^{(1)}$, we can expect the following property on our soliton cellular automaton.

- The internal degree of freedom of a soliton of length $l$ is described by the $U_{q}\left(A_{n-1}^{(1)}\right)$ crystal $B_{A}^{2, l}$.
- The exchange of the internal degree of freedom by the scattering of solitons of length
$l_{1}$ and $l_{2}\left(l_{1}>l_{2}\right)$ is given by the crystal isomorphism $B_{A}^{2, l_{2}} \otimes B_{A}^{2, l_{1}} \Im B_{A}^{2, l_{1}} \otimes B_{A}^{2, l_{2}}$.
- The phase shift of the scattering is described by the corresponding $H$ function.

We check these properties in this paper, thereby obtain our main theorem (Theorem 3.16).

The paper is organized as follows. In Sec. 2, we recapitulate necessary facts from the crystal theory. In Sec. 3, we construct conserved quantities. The main theorem is given in Sec. 3, where the scattering of solitons is studied.

## 2 Preliminaries

In this section we review some basic definitions and facts about crystals for the $U_{q}^{\prime}\left(D_{n}^{(1)}\right)$-crystal $B^{n, l}$ in Section 2.1. In order to describe the crystal graphs for the finite-dimensional modules of quantum groups of classical type, Kashiwara and Nakashima introduced the analogue of semi-standard tableaux, called KashiwaraNakashima (KN) tableaux [16].

### 2.1 Crystal $B^{n, 1}$

Crystal theory was introduced by Kashiwara [12] which provides a combinatorial way to study the representation theory of the quantum algebra $U_{q}(\mathfrak{g})$. In this paper $\mathfrak{g}=D_{n}^{(1)}$ is the corresponding quantum algebra. Let $P$ be the weight lattice, $\left\{\alpha_{i}\right\}_{0 \leq i \leq n}$ the simple roots, and $\left\{\Lambda_{i}\right\}_{0 \leq i \leq n}$ the fundamental weights of $D_{n}^{(1)}$. Let $\bar{\Lambda}_{i}$ denote the classical part of $\Lambda_{i}$. The crystal $B$ is a finite set with weight decomposition $B=\sqcup_{\lambda \in P} B_{\lambda}$. The Kashiwara operators $e_{i}, f_{i}(i=0,1, \cdots, n)$ act on $B$ as

$$
e_{i}: B_{\lambda} \longrightarrow B_{\lambda+\alpha_{i}} \sqcup\{0\}, \quad f_{i}: B_{\lambda} \longrightarrow B_{\lambda-\alpha_{i}} \sqcup\{0\} .
$$

These operators are nilpotent. By definition, we have $f_{i} b=b^{\prime}$ if and only if $b=e_{i} b^{\prime}$. Drawing $b \xrightarrow{i} b^{\prime}$ in such case, $B$ is endowed with the structure of colored oriented graph called crystal graph.

Let $\left\{\epsilon_{1}, \epsilon_{2}, \ldots, \epsilon_{n}\right\}$ orthonormal basis of the weight space of $D_{n}$. The simple roots and classical parts of fundamental weights for $D_{n}^{(1)}$ are expressed as

$$
\begin{aligned}
& \alpha_{0}=\delta-\epsilon_{1}-\epsilon_{2}, \quad \alpha_{n}=\epsilon_{n-1}+\epsilon_{n}, \quad \alpha_{i}=\epsilon_{i}-\epsilon_{i+1}, \quad \text { for } i=1,2, \ldots, n-1, \\
& \bar{\Lambda}_{n-1}=\frac{1}{2}\left(\epsilon_{1}+\cdots+\epsilon_{n-1}-\epsilon_{n}\right), \quad \bar{\Lambda}_{n}=\frac{1}{2}\left(\epsilon_{1}+\cdots+\epsilon_{n-1}+\epsilon_{n}\right) . \\
& \bar{\Lambda}_{i}=\epsilon_{1}+\epsilon_{2}+\cdots+\epsilon_{i} \quad \text { for } i=1,2, \ldots, n-2,
\end{aligned}
$$

We explain the Kirillov-Reshetikhin crystal $B^{n, l}, l \in \mathbb{Z}_{>0}$. We set $\mathfrak{A}=\{1,2, \cdots, n-1, n, \bar{n}, \overline{n-1}, \cdots, \overline{2}, \overline{1}\}$. The set of letters order on $\mathfrak{A}: 1 \prec 2 \prec \cdots \prec n-1 \prec{ }_{\bar{n}}^{n} \prec \overline{n-1} \prec \cdots \prec \overline{2} \prec \overline{1}$.
where there is no order between $n$ and $\bar{n}$. Then the crystal $B^{n, l}$ is given by

$$
B^{n, l}=\left\{\begin{array}{l|l|}
\hline c_{1} & c_{2}  \tag{2.2}\\
& c_{l} \\
\mid c_{j} \in B^{n, 1}, \text { and setting } \quad c_{j}=\begin{array}{|c|}
\hline c_{j n} \\
\hline \vdots \\
\hline c_{j 2} \\
\hline c_{j 1}
\end{array} & \begin{array}{l}
\text { where } c_{j p} \preceq c_{j+1, p} \\
\text { for } 1 \leq p \leq n, \\
1 \leq j<l
\end{array}
\end{array}\right\}
$$

The weight of $b \in B^{n, 1}$ is given by wt $b=\frac{1}{2} \sum_{j=1}^{n} \eta_{j} \epsilon_{j}$ where

$$
\eta_{j}= \begin{cases}+1 & \text { if } j \text { exist in } b \\ -1 & \text { if } \bar{j} \text { exist in } b\end{cases}
$$

### 2.2 Crystal structure on $B^{n, 1}$

For $i=0,1, \ldots, n$

$$
e_{i} b= \begin{cases}b^{\prime} & \text { if } w t b^{\prime}=w t b+\alpha_{i}(\bmod \mathbb{Z} \delta) \\ 0 & \text { if such } b^{\prime} \text { does not exist in } B^{n, 1}\end{cases}
$$

$$
f_{i} b= \begin{cases}b^{\prime \prime} & \text { if } w t b^{\prime \prime}=w t b-\alpha_{i}(\bmod \mathbb{Z} \delta) \\ 0 & \text { if such } b^{\prime \prime} \text { does not exist in } B^{n, 1}\end{cases}
$$

$B^{n, 1}$ is the crystal base [12] of the spin representation of the quantum affine algebra $U_{q}^{\prime}\left(D_{n}^{(1)}\right)$.

Example 2.1 When $n=4$, the crystal graph of $B^{4,1}$ is depicted as follows.


The crystal graph of $B^{1,1}$ is the same as above by interchanging the colors as $1 \leftrightarrow 4$.
Example 2.2 When $n=5$, the crystal graph of $B^{5,1}$ is depicted as follows.


We give the correspondence between the numbers in the crystal graph with our representation of crystal elements.


Example 2.3 When $n=6$, the crystal graph of $B^{6,1}$ is depicted as follows.


We give the correspondence between the numbers in the crystal graph with our representation of crystal elements.


$$
\begin{aligned}
& \begin{array}{ll}
\overline{1} & \overline{1} \\
\overline{2} & \overline{2}
\end{array} \\
& \boxed{31}=\begin{array}{l}
\overline{3} \\
\overline{4}
\end{array}, \quad \boxed{32}=\begin{array}{l}
\overline{3} \\
\overline{4}
\end{array} . \\
& 6 \quad \overline{5} \\
& 5 \quad \overline{6}
\end{aligned}
$$

Example 2.4 Consider the case $n=4$.

$$
\begin{gathered}
\text { Let } c_{0}=\begin{array}{c}
4 \\
3 \\
2 \\
1
\end{array}, c^{(1)}=\begin{array}{c}
\overline{3} \\
\overline{4} \\
2 \\
1
\end{array}, c^{(2)}=\begin{array}{c}
\overline{2} \\
\overline{4} \\
3 \\
1
\end{array}, c^{(3)}=\begin{array}{c}
\overline{1} \\
\overline{4} \\
3 \\
2
\end{array}, c^{(4)}=\begin{array}{c}
\overline{1} \\
\overline{3} \\
4 \\
2
\end{array}, c^{(5)}=\begin{array}{l}
\overline{1} \\
\overline{2} \\
4 \\
3
\end{array} . \\
e_{2}\left(c^{(2)}\right)=c^{(1)}, \quad f_{2}\left(c^{(2)}\right)=0, \quad f_{0}\left(c^{(2)}\right)=0 .
\end{gathered}
$$

Example 2.5 Consider the case $n=5$.


### 2.3 Crystal structure on $B^{n, l}$

Let $b \in B^{n, l}$.

$$
b=\begin{array}{|l|l|}
\hline c_{1} & c_{2}  \tag{2.3}\\
& \ldots \\
& c_{l} \\
& c_{j} \in B^{n, 1}
\end{array}
$$

The actions of $e_{i}, f_{i}$ for $i \neq 0$ can be calculated by using the rule called signature rule. For $b \in B^{n, l}$ we associate an element $c_{l} \otimes c_{l-1} \otimes \cdots \otimes c_{2} \otimes c_{1}$ of the tensor product $\left(B^{n, 1}\right)^{\otimes l}$ to find the indices $j, j^{\prime}$ such that

$$
\begin{gather*}
e_{i}\left(c_{l} \otimes c_{l-1} \otimes \cdots \otimes c_{2} \otimes c_{1}\right)=c_{l} \otimes c_{l-1} \otimes \cdots \otimes e_{i} c_{j} \otimes \cdots \otimes c_{2} \otimes c_{1}  \tag{2.4}\\
f_{i}\left(c_{l} \otimes c_{l-1} \otimes \cdots \otimes c_{2} \otimes c_{1}\right)=c_{l} \otimes c_{l-1} \otimes \cdots \otimes f_{i} c_{j^{\prime}} \otimes \cdots \otimes c_{2} \otimes c_{1} \tag{2.5}
\end{gather*}
$$

With this element we associate an $i$-signature : $\underbrace{-\ldots-}_{\varepsilon_{i}\left(c_{l}\right)} \underbrace{+\ldots+}_{\varphi_{i}\left(c_{l}\right)} \underbrace{-\ldots-}_{\varepsilon_{i}\left(c_{l-1}\right)} \underbrace{+\ldots+}_{\varphi_{i}\left(c_{l-1}\right)} \cdots \underbrace{-\ldots-}_{\varepsilon_{i}\left(c_{1}\right)} \underbrace{+\ldots+}_{\varphi_{i}\left(c_{1}\right)}$.

We then reduce the signature by deleting the adjacent +- pair successively. Eventually we obtain a reduced signature of the following form.

$$
--\cdots-++\cdots+
$$

Then the action $e_{i}$ (resp. $f_{i}$ ) corresponds to changing the rightmost - to + (resp. leftmost + to - ). If there is no $-($ resp. + ) in the signature, then the action of $e_{i}$ (resp. $f_{i}$ ) should be set to 0 . The value of $\varepsilon_{i}(b)$ (resp. $\varphi_{i}(b)$ ) is given by the number of $-($ resp. + ) in the reduced signature.

Example 2.6 Since the signature rule enables us to calculate the multiple tensor product of $B^{n, l}$ 's, we consider $B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. Let $c^{(j)}(j=1, \ldots, 5)$ as in Example 2.4.

Consider an element $b=\left(c_{0} c^{(1)} c^{(1)} c^{(2)}\right) \otimes\left(c_{0} c_{0} c^{(1)}\right) \otimes\left(c^{(1)} c^{(3)}\right) \in B^{4,4} \otimes B^{4,3} \otimes$ $B^{4,2}$. The 4-signature is given as follows
$b=\left(c^{(2)} \otimes c^{(1)} \otimes c^{(1)} \otimes c_{0}\right) \otimes\left(c^{(1)} \otimes c_{0} \otimes c_{0}\right) \otimes\left(c^{(3)} \otimes c^{(1)}\right)$
$\eta_{4}=\quad-\quad-\quad+\quad-\quad+\quad+$

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The reduced signature is $\eta_{4}=--+$, where the upper number signifies the component of the tensor product the sign belonged to. Therefore, we have
$e_{4} b=\left(c^{(2)} \otimes c^{(1)} \otimes e_{4}\left(c^{(1)}\right) \otimes c_{0}\right) \otimes\left(c^{(1)} \otimes c_{0} \otimes c_{0}\right) \otimes\left(c^{(3)} \otimes c^{(1)}\right)=\left(c_{0} c_{0} c^{(1)} c^{(2)}\right) \otimes$ $\left(c_{0} c_{0} c^{(1)}\right) \otimes\left(c^{(1)} c^{(3)}\right)$
$f_{4} b=\left(c^{(2)} \otimes c^{(1)} \otimes c^{(1)} \otimes c_{0}\right) \otimes\left(c^{(1)} \otimes f_{4}\left(c_{0}\right) \otimes c_{0}\right) \otimes\left(c^{(3)} \otimes c^{(1)}\right)=\left(c_{0} c^{(1)} c^{(1)} c^{(2)}\right) \otimes$ $\left(c_{0} c^{(1)} c^{(1)}\right) \otimes\left(c^{(1)} c^{(3)}\right)$

Example 2.7 Since the signature rule enables us to calculate the multiple tensor product of $B^{n, l}$, s, we consider $B^{4,4} \otimes B^{4,3} \otimes B^{4,2}$. Let $\tilde{c}^{(j)}(j=1, \ldots, 5)$ as in Example 2.5.

Consider an element $b^{\prime}=\left(\tilde{c}_{0} \tilde{c}^{(1)} \tilde{c}^{(1)} \tilde{c}^{(2)}\right) \otimes\left(\tilde{c}_{0} \tilde{c}_{0} \tilde{c}^{(1)}\right) \otimes\left(\tilde{c}^{(1)} \tilde{c}^{(3)}\right) \in B^{4,4} \otimes B^{4,3} \otimes$ $B^{4,2}$. The 5-signature is given as follows
$b^{\prime}=\left(\tilde{c}^{(2)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}_{0}\right) \otimes\left(\tilde{c}^{(1)} \otimes \tilde{c}_{0} \otimes \tilde{c}_{0}\right) \otimes\left(\tilde{c}^{(3)} \otimes \tilde{c}^{(1)}\right)$
$\eta_{5}=\quad-\quad-\quad+\quad-\quad+\quad+$

236
The reduced signature is $\eta_{5}=--+$, where the upper number signifies the component of the tensor product the sign belonged to. Therefore, we have
$e_{5} b^{\prime}=\left(\tilde{c}^{(2)} \otimes \tilde{c}^{(1)} \otimes e_{4}\left(\tilde{c}^{(1)}\right) \otimes \tilde{c}_{0}\right) \otimes\left(\tilde{c}^{(1)} \otimes \tilde{c}_{0} \otimes \tilde{c}_{0}\right) \otimes\left(\tilde{c}^{(3)} \otimes \tilde{c}^{(1)}\right)=\left(\tilde{c}_{0} \tilde{c}_{0} \tilde{c}^{(1)} \tilde{c}^{(2)}\right) \otimes$ $\left(\tilde{c}_{0} \quad \tilde{c}_{0} \tilde{c}^{(1)}\right) \otimes\left(\tilde{c}^{(1)} \tilde{c}^{(3)}\right)$
$f_{5} b^{\prime}=\left(\tilde{c}^{(2)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}^{(1)} \otimes \tilde{c}_{0}\right) \otimes\left(\tilde{c}^{(1)} \otimes f_{4}\left(\tilde{c}_{0}\right) \otimes \tilde{c}_{0}\right) \otimes\left(\tilde{c}^{(3)} \otimes \tilde{c}^{(1)}\right)=\left(\tilde{c}_{0} \tilde{c}^{(1)} \tilde{c}^{(1)} \tilde{c}^{(2)}\right) \otimes$ $\left(\tilde{c}_{0} \tilde{c}^{(1)} \tilde{c}^{(1)}\right) \otimes\left(\tilde{c}^{(1)} \tilde{c}^{(3)}\right)$

## $2.4 B^{n, l}$ and $B^{n-1, l}$ of type $D_{n}^{(1)}$

We give an affine crystal action on $B^{n, l}$. To do this we need $B^{n-1, l}$ of type $D_{n}^{(1)}$. $B^{n, l}$ and $B^{n-1, l}$ are associated to the spin nodes in the Dynkin diagram. As $\{1,2, \ldots, n\}$ crystals we have the isomorphisms

$$
\begin{equation*}
B^{n, l} \cong B\left(l \Lambda_{n}\right), \quad B^{n-1, l} \cong B\left(l \Lambda_{n-1}\right) \tag{2.6}
\end{equation*}
$$

To define the affine crystal action, we introduce an involution $\sigma: B^{n, l} \leftrightarrow B^{n-1, l}$ corresponding to the Dynkin diagram automorphism that interchanges the nodes $n$ to $n-1$. Let $J=\{2,3, \ldots, n\} . J$ is the $J$-highest if and only if $e_{i} b=0$ for every $i \in J$. By definition in [2], $\sigma$ is required to commute with $e_{i}, f_{i}(i \in J)$. Hence it sufficies to define $\sigma$ on $J$-highest elements in $B^{n, l}$ and all of the form of the LHS of (2.7) with some $a$, and mapped by $\sigma$ as

$$
\sigma: \underbrace{\begin{array}{cc}
n \cdots n & \overline{1} \cdots \overline{1}  \tag{2.7}\\
\vdots & \bar{n} \cdots \bar{n} \\
2 \cdots 2 & \vdots \\
1 \cdots 1 & 2 \cdots 2
\end{array}}_{a} \longleftrightarrow \underbrace{\underbrace{\begin{array}{cc}
\bar{n} \cdots \bar{n} & \overline{1} \cdots \overline{1} \\
\vdots & n \cdots n \\
2 \cdots 2 & \vdots \\
1 \cdots 1 & 2 \cdots 2
\end{array}}_{l-a}}_{l-a}
$$

Example 2.8 When $n=4$, the crystal graph of $B^{3,1}$ of type $D_{4}^{(1)}$ is depicted as follows.


Definition 2.9 The action $e_{0}$ and $f_{0}$ on $B^{n, l}$ is given by

$$
e_{0}=\sigma \circ e_{1} \circ \sigma, \quad f_{0}=\sigma \circ f_{1} \circ \sigma
$$

Example 2.10 Consider the case $B^{n, l}$ 's, where $n=4$ and $l=5$. Let $c^{(j)}(j=1, \ldots, 5)$ as in Example 2.4.

Consider an element $b=\left(c_{0} c_{0} c_{0} c^{(3)} c^{(3)}\right) \in B^{4,5}$. We are to calculate $e_{0} b$.

$$
\begin{align*}
& \begin{array}{|lllll|}
\overline{4} & \overline{4} & \overline{1} & \overline{1} & \overline{1} \\
3 & 3 & 4 & 4 & 4 \\
2 & 2 & 3 & 3 & 3 \\
1 & 1 & 2 & 2 & 2 \\
\hline
\end{array}  \tag{2.8}\\
& e_{1}\left|\begin{array}{ccccc}
\overline{4} & \overline{4} & \overline{1} & \overline{1} & \overline{1} \\
3 & 3 & 4 & 4 & 4 \\
2 & 2 & 3 & 3 & 3 \\
1 & 1 & 2 & 2 & 2
\end{array}\right|=\begin{array}{|ccccc}
\overline{4} & \overline{4} & \overline{2} & \overline{1} & \overline{1} \\
3 & 3 & 4 & 4 & 4 \\
2 & 2 & 3 & 3 & 3 \\
1 & 1 & 1 & 2 & 2 \\
\hline
\end{array} \tag{2.9}
\end{align*}
$$

$$
f_{1}\left|\begin{array}{ccccc}
\overline{4} & \overline{4} & \overline{1} & \overline{1} & \overline{1}  \tag{2.10}\\
3 & 3 & 4 & 4 & 4 \\
2 & 2 & 3 & 3 & 3 \\
1 & 1 & 2 & 2 & 2
\end{array}\right|=0
$$

But, $\sigma$ is required to commute with $e_{i}, f_{i}(i \in J)$. Then we apply $e_{3} e_{2}$ to get the $J$-highest.


### 2.5 Energy function

Next consider a $\mathbb{Z}$-valued function $H$ on $B \otimes B^{\prime}$ satisfying the following property: For any $b \in B, b^{\prime} \in B^{\prime}$ and $i$ such that $e_{i}\left(b \otimes b^{\prime}\right) \neq 0$

$$
H\left(e_{i}\left(b \otimes b^{\prime}\right)\right)= \begin{cases}H\left(b \otimes b^{\prime}\right)+1 & \text { if } i=0, \varphi_{0}(b) \geq \varepsilon_{0}\left(b^{\prime}\right), \varphi_{0}\left(\tilde{b^{\prime}}\right) \geq \varepsilon_{0}(\tilde{b}) \\ H\left(b \otimes b^{\prime}\right)-1 & \text { if } i=0, \varphi_{0}(b)<\varepsilon_{0}\left(b^{\prime}\right), \varphi_{0}\left(\tilde{b^{\prime}}\right)<\varepsilon_{0}(\tilde{b}) \\ H\left(b \otimes b^{\prime}\right) & \text { otherwise }\end{cases}
$$

$H$ is known to exist and unique up to additive constant. $\tilde{b}$ and $\tilde{b}^{\prime}$ are defined from the combinatorial $R$ matrix by $R\left(b \otimes b^{\prime}\right)=\tilde{b}^{\prime} \otimes \tilde{b}$. The existences of the isomorphism and energy function $H$ are guaranteed by the existence of the $R$ matrix. See Ref [13].

Note that we normalized $H$ so that we have $H\left(\left(c_{0}\right)^{l} \otimes\left(c_{0}\right)^{l^{\prime}}\right)=0$. Here and later, $\left(c_{0}\right)^{l}$ means $\underbrace{c_{0} \cdots c_{0}}_{l} \in B^{n, l}$.

Definition 2.11 A combinatorial $R$ matrix for the crystal $B^{n, s} \otimes B^{n, t}$ is a map

$$
R: B^{n, s} \otimes B^{n, t} \rightarrow B^{n, t} \otimes B^{n, s}
$$

satisfying
(1) $R \circ e_{i}=e_{i} \circ R, \quad R \circ f_{i}=f_{i} \circ R$ for $i=1, \ldots, n$, and
(2)



$c_{n^{\prime}}=. \quad$ (if $n$ is an odd number), $\quad c_{n^{\prime}}=. \quad$ (if $n$ is an even number), $\bar{n}$

$n^{\prime}=\left[\frac{n}{2}\right]$ and $t_{1}+t_{2}+\cdots+t_{n^{\prime}} \leq s, \quad s^{\prime}=s-\left(t_{1}+t_{2}+\cdots+t_{n^{\prime}}\right)$.
We explain how to calculate $R(b)$ for a general element $b$. Let $b \xrightarrow{e_{a_{1}}} b_{1} \xrightarrow{e_{a_{2}}}$ $b_{2} \cdots b_{m-2} \xrightarrow{e_{a_{m}}} b_{m-1} \xrightarrow{e_{a_{m}}} \tilde{b}$, where $e_{i}(\tilde{b})=0$ for every $i \neq 0$. Namely, $\tilde{b}$ is of the form


$$
R(b)=R\left(f_{a_{1}} \cdots f_{a_{m}} \tilde{b}\right)=f_{a_{1}} \cdots f_{a_{m}} R(\tilde{b})
$$

## Example 2.12

Set $\left.\left.b=\begin{array}{lll}\overline{3} & \overline{3} & \overline{1} \\ \overline{4} & \overline{4} & \overline{3} \\ 2 & 2 & 4 \\ 1 & 1 & 2\end{array}\right] \otimes \begin{array}{|cc|}\hline \overline{2} & \overline{2} \\ \overline{4} & \overline{3} \\ 3 & 4 \\ 1 & 1\end{array}\right]$ and we calculate $R(b)$.
$\left.\left.\tilde{b}=e_{4} e_{2}{ }^{3} e_{4}{ }^{2} e_{3}{ }^{2} e_{1} b=\begin{array}{|ccc|}\hline & 4 & 4 \\
3 & 3 & 3 \\
2 & 2 & 2 \\
1 & 1 & 1\end{array} \right\rvert\, \otimes \begin{array}{|cc|}\hline \overline{3} & \overline{3} \\
\overline{4} & \overline{4} \\
2 & 2 \\
1 & 1\end{array}\right]$ and \(\left.R(\tilde{b})=\begin{array}{|lll}4 \& 4 <br>
3 \& 3 <br>
2 \& 2 <br>

1 \& 1\end{array}\right] \otimes\)| 4 | $\overline{3}$ | $\overline{3}$ |
| :---: | :---: | :---: |
| 3 | $\overline{4}$ | $\overline{4}$ |
| 2 | 2 | 2 |
| 1 | 1 | 1 | .

Since $\left.\left.f_{1} f_{3}{ }^{2} f_{4}{ }^{2} f_{2}{ }^{3} f_{4} R(\tilde{b})=\begin{array}{|cc|}\hline \overline{3} & \overline{3} \\ \overline{4} & \overline{4} \\ 2 & 2 \\ 1 & 1\end{array} \right\rvert\, \otimes \begin{array}{|ccc|}\hline \overline{2} & \overline{2} & \overline{1} \\ \overline{4} & \overline{3} & \overline{3} \\ 3 & 4 & 4 \\ 1 & 1 & 2\end{array}\right]$,
we have $\left.\left.R\left(\left.\begin{array}{lll}\overline{3} & \overline{3} & \overline{1} \\ \overline{4} & \overline{4} & \overline{3} \\ 2 & 2 & 4 \\ 1 & 1 & 2\end{array} \right\rvert\, \otimes \begin{array}{|ccc}\overline{2} & \overline{2} \\ \overline{4} & \overline{3} \\ 3 & 4 \\ 1 & 1\end{array}\right)=\begin{array}{|lll}\overline{3} & \overline{3} \\ \overline{4} & \overline{4} \\ 2 & 2 \\ 1 & 1\end{array} \right\rvert\, \otimes \begin{array}{|ccc|}\hline \overline{2} & \overline{2} & \overline{1} \\ \overline{4} & \overline{3} & \overline{3} \\ 3 & 4 & 4 \\ 1 & 1 & 2\end{array}\right]$.

Lemma 2.13 We give all the images of the isomorphism $B^{n, l} \otimes B^{n, 1} \simeq B^{n, 1} \otimes B^{n, l}$ that will be needed in the next section. Let the symbol

$$
\begin{gathered}
\beta \\
\alpha+\alpha^{\prime}
\end{gathered}
$$

signify $\alpha \otimes \beta \mapsto \beta^{\prime} \otimes \alpha^{\prime}$ under the isomorphism. We also set

and $c_{0}$ is the column of height $n$ without barred letters.

We list up the cases as below where $i, j, k, l, m, p \in \mathbb{Z}_{\geq 0}$.

## Case 1.

(i) $m>0$
(ii) $l>0$
$c_{0}^{i} c_{a}^{j} c_{b}^{k} c_{d}^{l} c_{f}^{m} \stackrel{c_{0}}{+} c_{f}^{+i+1} c_{a}^{j} c_{b}^{k} c_{d}^{l} c_{f}^{m-1}$

(iii) $k>0$
(iv) $j>0$
(v) $i>0$
$c_{0}^{i} c_{a}^{j} c_{b}^{k}+c_{0}^{i+1} c_{a}^{j} c_{b}^{k-1}$
$c_{b}$
$c_{0}^{i} c_{a}^{j}+c_{0}^{i+1} c_{a}^{j-1}$
$c_{a}$
$c_{0}$
$c_{0}^{i}+c_{0}^{i}$
$c_{0}$

## Case 2.

(i) $i>0, m>0$
(ii) $m>0$
$c_{0}^{i} c_{a}^{j} c_{b}^{k} c_{d}^{l} c_{f}^{m}{ }_{c_{a}}^{c_{a}} c_{0}^{i-1} c_{a}^{j+1} c_{b}^{k} c_{d}^{l+1} c_{f}^{m-1}$
(iii) $i>0$
$c_{0}^{i} c_{a}^{j} c_{b}^{k} c_{d}^{l} \stackrel{c_{a}}{+} c_{0}^{i-1} c_{a}^{j+1} c_{b}^{k} c_{d}^{l}$
(v) $k>0$
$c_{a}^{j} c_{b}^{k} \stackrel{c_{a}}{+} c_{b}^{j+1} c_{b}^{k-1}$
(iv) $l>0$
$c_{a}^{j} c_{b}^{k} c_{d}^{l} \stackrel{c_{a}}{+c_{d}} c_{a}^{j+1} c_{b}^{k} c_{d}^{l-1}$

Case 3.
(i) $j>0, l>0$
$c_{0}^{i} c_{a}^{j} c_{b}^{k} c_{d}^{l} c_{f}^{m} \stackrel{c_{b}}{+} c_{0}^{i} c_{a}^{j-1} c_{b}^{k+1} c_{d}^{l-1} c_{e} c_{f}^{m}$
(iii) $p>0$
$c_{0}^{i} c_{b}^{k} c_{d}^{l} c_{e}^{p} c_{f}^{m} \underset{c_{e}}{c_{b}} c_{0}^{i} c_{b}^{k+1} c_{d}^{l} c_{e}^{p-1} c_{f}^{m}$
(v) $m>0, \quad i>0$
$c_{0}^{i} c_{b}^{k} c_{d}^{l} c_{f}^{m} \stackrel{c_{b}}{c_{a}} c_{0}^{i-1} c_{b}^{k+1} c_{d}^{l+1} c_{f}^{m-1}$
(vii) $m>0$
$c_{b}^{k} c_{d}^{l} c_{f}^{m} \underset{c_{f}}{c_{b}} c_{b}^{k+1} c_{d}^{l} c_{f}^{m-1}$
(vi) $l>0$
$c_{b}^{k} c_{e}^{p} c_{d}^{l} \stackrel{c_{b}}{{ }_{c}} c_{d}^{k+1} c_{e}^{p} c_{d}^{l-1}$
(viii) $i>0$
(ii) $j>0$

$$
c_{0}^{i} c_{a}^{j} c_{b}^{k} c_{e}^{p} c_{f}^{m} \stackrel{c_{b}}{c_{a}} c_{0}^{i} c_{a}^{j-1} c_{b}^{k+1} c_{e}^{p} c_{f}^{m}
$$

(iv) $i>0$

$$
c_{0}^{i} c_{b}^{k} c_{e}^{p} c_{f}^{m} \stackrel{c_{b}}{c_{a}} c_{0}^{i-1} c_{b}^{k+1} c_{d} c_{e}^{p}
$$

$$
c_{b}^{k} c_{e}^{p} c_{d}^{l} \underset{c_{d}}{+} c_{b}^{k+1} c_{e}^{p} c_{d}^{l-1}
$$

$$
c_{0}^{i} c_{b}^{k} c_{d}^{l} \stackrel{c_{b}}{\underset{c_{0}}{+}} c_{0}^{i-1} c_{b}^{k+1} c_{d}^{l}
$$

(ix) $p>0$
$(\mathrm{x}) l>0$
(xi) $k>0$

$$
c_{b}^{k} c_{e}^{p} \stackrel{c_{b}}{c_{e}} c_{b}^{k+1} c_{e}^{p-1}
$$

$c_{b}^{k} c_{d}^{l} \stackrel{c_{b}}{+} c_{b}^{k+1} c_{d}^{l-1}$ $c_{b}^{k} \stackrel{c_{b}}{+} c_{b}^{k}$

## Case 4.

(i) $j>0, l>0$
(ii) $j>0, k>0$
$c_{0}^{i} c_{a}^{j} c_{b}^{k} c_{d}^{l} c_{f}^{m} \underset{c_{d}}{c_{d}} c_{0}^{i+1} c_{a}^{j-1} c_{b}^{k} c_{d}^{l-1} c_{f}^{m+1}$
$c_{0}^{i} c_{a}^{j} c_{b}^{k} c_{f}^{m} \stackrel{c_{d}}{\text { 世 }_{b}} c_{0}^{i+1} c_{a}^{j-1} c_{b}^{k-1} c_{f}^{m+1}$
(iii) $k>0$
$c_{0}^{i} c_{b}^{k} c_{d}^{l} c_{f}^{m} \underset{c_{b}}{c_{d}} c_{0}^{i} c_{b}^{k-1} c_{d}^{l+1} c_{f}^{m}$
(iv) $j>0$
$c_{0}^{i} c_{a}^{j} c_{f}^{m} \stackrel{c_{d}}{c_{a}} c_{0}^{i+1} c_{a}^{j-2} c_{f}^{m+1}$
(v) $i>0, \quad j>0, \quad m>0$.
$c_{0}^{i} c_{d}^{l} c_{f}^{m}+c_{d}^{+1-1} c_{a}^{j-1} c_{d}^{3} c_{f}^{m-1}$
$c_{a}$
(vi) $k>0$
$c_{0}^{i} c_{b}^{k} c_{f}^{m} \stackrel{c_{d}}{+} c_{0}^{i} c_{b}^{k-1} c_{d} c_{f}^{m}$
(vii) $i>0, m>0$
(viii) $l>0, \quad j>0$
(ix) $i>0$

$$
\begin{aligned}
& c_{0}^{i} c_{d}^{l} c_{f}^{m} \stackrel{c_{d}}{c_{a}} c_{0}^{i-1} c_{d}^{l+2} c_{f}^{m-1} \\
& c_{0}^{i} c_{a}^{j} c_{d}^{l} \stackrel{c_{d}}{c_{d}} c_{0}^{i+1} c_{a}^{j-1} c_{d}^{l-1} c_{f} \\
& c_{0}^{i} c_{d}^{l} \stackrel{c_{d}}{{ }_{c}} c_{0}^{i-1} c_{d}^{l+1} \\
& \text { (x) } j>0, l>0 \\
& c_{a}^{j} c_{d}^{l} c_{f}^{m} \stackrel{c_{d}}{c_{d}} c_{0}^{i} c_{a}^{j-1} c_{d}^{l-1} c_{f}^{m+1} \\
& \text { (xi) } i>0, \quad m>0 \\
& c_{0}^{i} c_{f}^{m} \stackrel{c_{d}}{c_{a}} c_{0}^{i-1} c_{d}^{2} c_{f}^{m-1} \\
& \text { (ivx) } m>0 \\
& \text { (xiii) } i>0, m>0 \\
& c_{0}^{i} c_{d}^{l} c_{f}^{m} \stackrel{c_{d}}{c_{a}} c_{0}^{i-1} c_{d}^{l+2} c_{f}^{m-1} \\
& c_{d}^{l} c_{f}^{m} \stackrel{c_{d}}{c_{f}} c_{d}^{l+1} c_{f}^{m-1} \\
& \text { (ix) } l>0 \\
& c_{d}^{l} \stackrel{c_{d}}{{ }_{c}} c_{d}^{l}
\end{aligned}
$$

## Case 5.

(i) $i>0$
(ii) $l>0$
$c_{0}^{i} c_{a}^{j} c_{b}^{k} c_{d}^{l} c_{f}^{m}{\stackrel{c_{f}}{+}}_{c_{0}}^{+} c_{0}^{i-1} c_{a}^{j} c_{b}^{k} c_{d}^{l} c_{f}^{m+1}$

$$
c_{a}^{j} c_{b}^{k} c_{d}^{l} c_{f}^{m} \stackrel{c_{f}}{c_{d}} c_{a}^{j} c_{b}^{k} c_{d}^{l-1} c_{f}^{m+1}
$$

(iii) $k>0$
(iv) $j>0$
(v) $m>0$

$$
c_{a}^{j} c_{b}^{k} c_{f}^{m} \underset{c_{b}}{c_{f}}+c_{a}^{j} c_{b}^{k-1} c_{f}^{m+1}
$$

$$
\begin{gathered}
c_{a}^{j} c_{f}^{m}+c_{a}^{j-1} c_{f}^{m+1} \\
c_{a}
\end{gathered}
$$

$$
c_{f}^{m} \stackrel{c_{f}}{+} c_{f}^{m}
$$

Proof for Case 1 where $k>0$.
$\left(c_{b}^{k} \otimes c_{a}^{j} \otimes c_{0}^{i}\right) \otimes c_{0} \xrightarrow{e_{n-2}^{k}}\left(c_{a}^{j+k} \otimes c_{0}^{i}\right) \otimes c_{0} \xrightarrow{e_{n}^{j+k}} c_{0}^{i+j+k} \otimes c_{0} . \quad$ Reform $c_{0}^{i+j+k} \otimes c_{0}$ to $c_{0} \otimes\left(c_{0} \otimes c_{0}^{i+j+k}\right)$. Then $c_{0} \otimes c_{0}^{i+j+k} \xrightarrow{f_{n}^{j+k}} c_{a} \otimes\left(c_{a}^{j+k-1} \otimes c_{0}^{i+1}\right) \xrightarrow{f_{n-2}^{k}} c_{b} \otimes\left(c_{b}^{k-1} \otimes c_{a}^{j} \otimes c_{0}^{i+1}\right)$. Hence $R\left(c_{0}^{i} c_{a}^{j} c_{b}^{k} \otimes c_{0}\right)=c_{b} \otimes c_{0}^{i+1} c_{a}^{j} c_{b}^{k-1}$.

Definition 2.14 $A$ combinatorial $R$ matrix for the crystal $B \otimes B^{\prime}$ is a map $R$ : $A f f(B) \otimes \operatorname{Aff}\left(B^{\prime}\right) \rightarrow \operatorname{Aff}\left(B^{\prime}\right) \otimes A f f(B)$ given by

$$
R\left(z^{d} b \otimes z^{d^{\prime}} b^{\prime}\right)=z^{d^{\prime}+H\left(b \otimes b^{\prime}\right)} \tilde{b^{\prime}} \otimes z^{d-H\left(b \otimes b^{\prime}\right)} \tilde{b}
$$

where $b \otimes b^{\prime} \mapsto \tilde{b}^{\prime} \otimes \tilde{b}$ under the isomorphism $B \otimes B^{\prime} \xrightarrow{\sim} B^{\prime} \otimes B$. The following result is a direct consequence of the ordinary (i.e. not combinatorial) Yang-Baxter equation.

### 2.6 Yang-Baxter equation

Let us define the affinization $\operatorname{Aff}(B)$ of the crystal $B$. We introduce an indeterminate $z$ ( the spectral parameter) and set

$$
\operatorname{Aff}(B)=\left\{z^{d} b \mid d \in \mathbb{Z}, b \in B\right\}
$$

Thus $\operatorname{Aff}(B)$ is an infinite set. $z^{0} b \in \operatorname{Aff}(B)$ will often be written as $b . \operatorname{Aff}(B)$ also admits the crystal structure by $e_{i} . z^{d} b=z^{d+\delta_{i 0}}\left(e_{i} b\right), f_{i} \cdot z^{d} b=z^{d-\delta_{i 0}}\left(f_{i} b\right)$.

Proposition 2.15 (Yang-Baxter equation). Let $B^{n, l}=B_{l}$. The following equation hold on $\operatorname{Aff}\left(B_{l}\right) \otimes \operatorname{Aff}\left(B_{l^{\prime}}\right) \otimes \operatorname{Aff}\left(B_{l^{\prime \prime}}\right)$.

$$
(R \otimes 1)(1 \otimes R)(R \otimes 1)=(1 \otimes R)(R \otimes 1)(1 \otimes R)
$$

## 3 Soliton cellular automata

### 3.1 States and time evolutions.

Consider the crystal $\left(B^{n, 1}\right)^{\otimes N}$ for sufficiently large $N$. The elements of $\left(B^{n, 1}\right)^{\otimes N}$ we have in mind are of the following form:
$\cdots \otimes c_{0} \otimes \cdots \otimes c_{0} \otimes c_{1} \otimes \cdots \otimes c_{l} \otimes c_{0} \otimes \cdots \otimes c_{0} \otimes \cdots$,

Namely, relatively few elements are non $c_{0}$, and almost all are $c_{0}$. In the assertions below, we embed, if necessary, $\left(B^{n, 1}\right)^{\otimes N}$ into $\left(B^{n, 1}\right)^{\otimes N^{\prime}}\left(N<N^{\prime}\right)$ by

$$
\begin{aligned}
& \left(B^{n, 1}\right)^{\otimes N} \hookrightarrow\left(B^{n, 1}\right)^{\otimes N^{\prime}}, \\
b_{1} \otimes \cdots \otimes b_{N} \mapsto & b_{1} \otimes \cdots \otimes b_{N} \otimes \underbrace{c_{0} \otimes \cdots \otimes c_{0}}_{N^{\prime}-N} .
\end{aligned}
$$

Lemma 3.1 By iterating $B^{n, l} \otimes B^{n, 1} \rightarrow B^{n, 1} \otimes B^{n, l}$ we consider a map

$$
\begin{gathered}
B^{n, l} \otimes B^{n, 1} \otimes \cdots \otimes B^{n, 1} \xrightarrow{\sim} B^{n, 1} \otimes \cdots \otimes B^{n, 1} \otimes B^{n, l} \\
\left(c_{0}\right)^{l} \otimes b_{1} \otimes \cdots \otimes b_{N} \mapsto \tilde{b}_{1} \otimes \cdots \otimes \tilde{b}_{N} \otimes \tilde{b}
\end{gathered}
$$

then there exists an integer $N_{0}$ such that $\tilde{b}=\left(c_{0}\right)^{l}$ for $N \geq N_{0}$.

Taking sufficiently large $N$ such that the above lemma holds, we define a map $T_{l}:\left(B^{n, 1}\right)^{\otimes N} \longrightarrow\left(B^{n, 1}\right)^{\otimes N}$ by $b_{1} \otimes \cdots \otimes b_{N} \mapsto \tilde{b}_{1} \otimes \cdots \otimes \tilde{b}_{N}$.

Lemma 3.2 For a fixed element of $\left(B^{n, 1}\right)^{\otimes N}$ as a Lemma 3.1, there exists an integer $l_{0}$ such that $T_{l}=T_{l_{0}}$ for any $l \geq l_{0}$.

Both lemmas are obvious from Lemma 2.13.

An element of $\left(B^{n, 1}\right)^{\otimes N}$ having the property described in the beginning of this subsection will be called a state. Lemma 3.1 and Lemma 3.2 enable us to define an operator $T=\lim _{l \rightarrow \infty} T_{l}$ on the space of states. Application of $T$ induces a transition of state. Thus it can be regarded as a certain dynamical system, in which $T$ plays the role of 'time evolution'. By the same reason, $T_{l}$ may also be viewed as another time evolution. (In this paper, time evolution means the one by $T$ unless otherwise stated.)

### 3.2 Conservation laws

Fix sufficiently large $N$ and consider a composition of the combinatorial $R$ matrices

$$
R_{l}=R_{N N+1} \cdots R_{23} R_{12}: \operatorname{Aff}\left(B^{n, l}\right) \otimes \operatorname{Aff}\left(B^{n, 1}\right)^{\otimes N} \rightarrow \operatorname{Aff}\left(B^{n, 1}\right)^{\otimes N} \otimes \operatorname{Aff}\left(B^{n, l}\right)
$$

Here $R_{i i+1}$ signifies that the $R$ matrix acts on the $i$-th and $(i+1)$-th components of the tensor product. Applying $R_{l}$ to an element $\left(c_{0}\right)^{l} \otimes p\left(p=b_{1} \otimes \cdots \otimes b_{N}\right)$, we have

$$
\begin{gathered}
R_{l}\left(\left(c_{0}\right)^{l} \otimes p\right)=z^{H_{1}} \tilde{b}_{1} \otimes z^{H_{2}} \tilde{b}_{2} \otimes \cdots \otimes z^{H_{N}} \tilde{b}_{N} \otimes z^{\tilde{E}_{l}(p)}\left(c_{0}\right)^{l} \\
\tilde{E}_{l}(p)=-\sum_{j=1}^{N} H_{j}, \quad H_{j}=H\left(b^{(j-1)} \otimes b_{j}\right)
\end{gathered}
$$

where $b^{(0)}=\left(c_{0}\right)^{l}$ and $b^{(j)}(1 \leq j<N)$ is defined by

$$
\begin{gathered}
B^{n, l} \otimes \underbrace{B^{n, 1} \otimes \cdots \otimes B^{n, 1}}_{j} \simeq \underbrace{B^{n, 1} \otimes \cdots \otimes B^{n, 1}}_{j} \otimes B^{n, l} \\
\left(c_{0}\right)^{l} \otimes b_{1} \otimes \cdots \otimes b_{j} \mapsto \tilde{b}_{1} \otimes \cdots \otimes \tilde{b}_{j} \otimes b^{(j)}
\end{gathered}
$$

Lemma 3.3 Let $H$ be the energy function and $-H_{j}=-H\left(b^{(j-1)} \otimes b_{j}\right) \in\left\{0,1,2, \ldots, n^{\prime}\right\}$. Then $b^{(j-1)} \otimes b_{j}$ commute with $e_{i}, f_{i}$ until it will get in form $c_{0}^{i} c_{d}^{l} \otimes c_{d} \mapsto c_{0} \otimes c_{0}^{i-1} c_{d}^{l+1}$. as in Case 4 in Lemma 2.13. Then by using the rule in [24] we will get $-H_{j}=1$ if and only if $c_{0}^{i} c_{d}^{l} \otimes c_{d} \mapsto c_{0} \otimes c_{0}^{i-1} c_{d}^{l+1}$.
Proof. Use the definition of energy function in [31] to prove it directly.

Proposition 3.4 For an element $p \in\left(B^{n, 1}\right)^{\otimes N}$, we have
(1) $T_{l} T_{l^{\prime}}(p)=T_{l^{\prime}} T_{l}(p)$.
(2) $E_{l}\left(T_{l^{\prime}}(p)\right)=E_{l}(p)$. In particular, $E_{l}(T(p))=E_{l}(p)$.

We refer to [3] for the proof.

### 3.3 Soliton

A state of the following form is called an $m$ soliton state of length $l_{1}, l_{2}, \ldots, l_{m}$,

$$
\begin{equation*}
\ldots\left[l_{1}\right] \ldots \ldots .\left[l_{2}\right] \ldots . . . \ldots . .\left[l_{m}\right] \ldots \ldots \ldots \tag{3.1}
\end{equation*}
$$

Here $\cdots[l] \cdots$ denotes a local configuration such as
$\cdots \otimes c_{0} \otimes c_{0} \otimes \underbrace{c_{l} \otimes c_{l-1} \otimes \cdots \otimes c_{1}}_{l} \otimes c_{0} \otimes c_{0} \otimes \cdots, \quad\left(c_{0} \preceq c_{1} \preceq c_{2} \preceq \cdots \preceq c_{l}\right)$
where $c_{0}$ means column without barred letter and $c_{j} \in B^{n, 1}$ with exactly two barred letters. That means we cannot form the soliton as above when the number of barred letter is more than two. $c \preceq c^{\prime}$ if and only if $a_{j}(j$-th entry of $c), a_{j} \preceq a_{j}^{\prime}$ for all $j=1,2, \ldots, n$.

Remark. It would be an interesting problem to consider color separation scheme in [[34]: §4.7]. A reasonable choice of $B_{\natural}$ such as $B^{n-1,1}$ or $B^{n-2,1}$ seems to fail for $n=4$.

Lemma 3.5 Let $p$ be a one-soliton state of length $l$, then
(1) The $k$ th conserved quantity of $p$ is given by $E_{k}(p)=\min (k, l)$.
(2) The state $T_{k}(p)$ is obtained by the rightward shift by $E_{k}(p)$ lattice steps.

Proof. (1) Recall that the conserved quantity $E_{k}$ is a sum of local $H$ functions $-H_{j}=-H\left(b^{(j-1)} \otimes b_{j}\right) . b^{(j-1)} \otimes b_{j} \mapsto \tilde{b}_{j} \otimes b^{(j)}$ commutes with $f_{i}(i \in 1,2, \ldots, n-1)$ until we will get the lowest weight where $f_{i} b=0$ for any $i \in\{1,2, \ldots, n-1\}$. Then we apply $f_{0}, e_{0}$ and we will get $\left(c_{0}\right)^{k} \otimes c_{0} \mapsto c_{0} \otimes\left(c_{0}\right)^{k}$. By using Lemma 3.3, $-H_{j}=1$ if and only if $\left(c_{0}\right)^{i} c_{d}^{l} \otimes c_{d} \mapsto c_{0} \otimes\left(c_{0}\right)^{i-1} c_{d}^{l+1} . E_{k}(p)$ is the sum of the local $-H_{j}=1$ in the tensor product in $B^{n, l}$. Hence $E_{k}=\min (k, l)$. Similarly, the statement (2) follows from the rule (2.3) and Lemma 3.1.

Definition 3.6 For any state $p$, the number $N_{l}=N_{l}(p)(l=1,2, \ldots)$ are defined by

$$
E_{l}=\sum_{k \geq 1} \min (k, l) N_{k}, \quad E_{0}=0, \quad N_{i}=-E_{l-1}+2 E_{l}-E_{l+1}
$$

By Lemma 3.5, we have
Proposition 3.7 For m-soliton state (3.1), $N_{l}$ is the number of solitons of length $l$,

$$
N_{l}=\#\left\{j \mid l_{j}=l\right\}
$$

This proposition implies the stability of solitons, since the number $E_{l}(p)$, and hence $N_{l}(p)$, are conserved.

### 3.4 Type A $B^{r, s}$

In this subsection, we recall the crystal structure of $B^{r, s}$ for arbitrary $r, s$ and the combinatorial $R$ for $B^{r, s} \otimes B^{r^{\prime}, s^{\prime}}$. Our reference is [31]. We use the French notation for semistandard tableau, which is upside-down from [31]

Our $U_{q}^{\prime}\left(A_{n-1}^{(1)}\right)$-crystal $B^{r, s}\left(1 \leq r \leq n-1, s \in \mathbb{Z}_{>0}\right)$ is as a set, identified with the set of semistandard tableau of rectangular shape $\left(s^{r}\right)$ with letters from $\{1,2, \ldots, n\}$. For an element $t$ of $B^{r, s}$, let $t_{i j}$ denote the letter in the $i$-th row from bottom and $j$-th column of $t$. We first describe the action of $e_{i}, f_{i}$ for $i=1,2, \ldots, n-1$. For this purpose, let us define the Japanese reading word of $t$ by

$$
J(t)=w^{(1)} w^{(2)} \cdots w^{(s)}, \quad w^{(j)}=t_{1 j} t_{2 j} \cdots t_{r j} \quad(j=1,2, \ldots, s)
$$

We then regard $J(t)$ as an element of $\left(B^{1,1}\right)^{\otimes(r s)}$. Namely, each letter is considered to be an element of $B^{1,1}$.

The bumping algorithm is defined for a pair of tableau $t$ and single word $u$ and depicted as $t \leftarrow u$. First, let us consider the case where $t$ is one-row tableau. If $t$ is empty, $t \leftarrow u$ is defined to be the tableau $u$ with one node. Otherwise, let $t=t_{11} t_{12} \cdots t_{1 m}$ and look at

$$
t_{11} t_{12} \cdots t_{1 m} \leftarrow u
$$

If $t_{1 m} \leq u$, then define

$$
t \leftarrow u=t_{11} t_{12} \cdots t_{1 m} u
$$

and the algorithm stops (case (a)). Otherwise, set $i_{1}=\min \left\{i \mid t_{1 i}>u\right\}$ and define

$$
t \leftarrow u=t_{11} \cdots t_{1 i_{1}-1} u t_{1 i_{1}+1} \cdots t_{1 m}
$$

and we have the single word $t_{1 i_{1}}$ bumped out from $t$ (case (b)). Now suppose we have a tableau $t$ of $l$ rows and let $t_{i}$ be the $i$-th row of $t$. The bumping algorithm $t \leftarrow u$ proceeds as follows. Set $t_{1}^{\prime}=t_{1} \leftarrow u$. If case (a) occurs, the algorithm stops. Otherwise, let $u_{1}$ be the letter bumped out and set $t_{2}^{\prime}=t_{2} \leftarrow u_{1}$. We again devide the algorithm into the two cases. The algorithm proceeds until it stops. If case (b) still occurs in the highest row, we append the empty row above it.

Example 3.8 Let

$$
t=\begin{aligned}
& 23444 \\
& 11122
\end{aligned}
$$

and $u=1$. The bumping algorithm proceeds as follows.

$$
\begin{array}{lll}
23444 \\
11122 & \leftarrow 1 & 23444 \\
11112
\end{array} \leftarrow 2
$$

And we have the answer.

Example 3.9 Let

$$
t=\begin{aligned}
& 24555 \\
& 11334
\end{aligned}
$$

and $u=2$. The bumping algorithm proceeds as follows.

$$
\begin{array}{lll}
24555 \\
11334 & \leftarrow 2 & 24555 \\
11234
\end{array}
$$

And we have the answer.
4
23555
11234

For a tableau $t \in B^{r, s}$ we define the row word row $t$ by

$$
\operatorname{row}(t)=t_{r} t_{r-1} \cdots t_{1} \quad t_{i}=t_{i 1} t_{i 2} \cdots t_{i s} \quad(i=1,2, \ldots, r)
$$

Let $t$ be a tableau and $w=u_{1} u_{2} \cdots u_{l}$ a word of length $l$. Let $t \leftarrow w$ be a tableau obtained by applying the bumping algorithm for a single word $u_{j}$ successively as

$$
\left(\cdots\left(\left(t \leftarrow u_{1}\right) \leftarrow u_{2}\right) \leftarrow \cdots\right) \leftarrow u_{l}
$$

Then we have the following proposition to obtain the combinatorial $R$ for $B^{r, s} \otimes B^{r, m}$.

Proposition 3.10 [31] Assume $t \in B^{r, m}$ and $t^{\prime} \in B^{r, s}$. Then $t^{\prime} \otimes t$ is mapped to $\tilde{t} \otimes \tilde{t}^{\prime}$ by the crystal isomorphism

$$
B^{r, s} \otimes B^{r, m} \longrightarrow B^{r, m} \otimes B^{r, s} \quad \text { if and only if } \quad t \leftarrow \operatorname{row}\left(t^{\prime}\right)=\tilde{t}^{\prime} \leftarrow \operatorname{row}(\tilde{t})
$$

Moreover, the energy function $H\left(t^{\prime} \otimes t\right)$ is given by the number of nodes in the shape of $t \leftarrow \operatorname{row}\left(t^{\prime}\right)$ that are strictly north of the $r$-th row.

Note that the decomposition of $B^{r, s} \otimes B^{r, m}$ into $U_{q}\left(A_{n-1}^{(1)}\right)$-crystals is multiplicity free. From this fact, it follows that for a given pair $t^{\prime} \otimes t$ we can determine $\tilde{t}, \tilde{t}^{\prime}$ uniquely. To explain the algorithm of computing $\tilde{t}, \tilde{t}^{\prime}$ we prepare terminology. Let $\theta$ be a skew tableau, that is, set-theoretical difference of a Young diagram from a smaller one with letters in each node. Let $\tau$ be the shape of $\theta . \theta$ is called a vertical $k$-strip if $|\tau|=k$ and $\tau_{i} \leq 1$ for any $i \geq 1$. The algorithm to obtain $\tilde{t}, \tilde{t}^{\prime}$ is given as follows. Let $p$ be the tableau obtained by the bumping algorithm $t \leftarrow \operatorname{row}\left(t^{\prime}\right)$. We attach an integer from 1 to $r m$ to each node of the skew tableau $p-p^{\prime}$, where $p^{\prime}$ is the NE part of $p$ whose shape is $\left(s^{r}\right)$. The integer should be labeled in the following manner. Let $\theta_{1}$ be the rightmost vertical $r$-strip in $p-p^{\prime}$ as lower as possible. We attach integers 1 through $r$ from lower nodes. Remove $\theta_{1}$ from $p-p^{\prime}$ and define the vertical $r$-strip $\theta_{2}$ in similar manner. Continue it until we finish attaching all integers up to rm . Next we apply the reverse bumping algorithm according to the order of the labeling. Namely we find a word $u_{1}$ and a tableau $p_{1}$ whose shape is (shape of $p$ )-(node of label 1 ), such that $p_{1} \leftarrow u_{1}=p$. (Note that such a pair $p_{1}, u_{1}$ ) is unique.) We repeat this procedure to obtain $u_{2}$ and $p_{2}$ by replacing $p$ and the node of label 1 with $p_{1}$ and the node of label 2 and continue until we arrive at a tableau of shape $\left(s^{r}\right)$. Then we have

$$
\tilde{t}=\left(\left(\cdots\left(\phi \leftarrow u_{r m}\right) \leftarrow \cdots\right) \leftarrow u_{2}\right) \leftarrow u_{1} \text { and } \tilde{t^{\prime}}=p_{r m} .
$$

Note that in [31], the energy function $H\left(t^{\prime} \otimes t\right)$ is given by the number of nodes in the shape of $t^{\prime} \leftarrow \operatorname{row}(t)$ that are strictly east of the $\max (m, s)$-th column.

We introduce $\nu$ as a map sending an element $b$ of the $U_{q}^{\prime}\left(D_{n}^{(1)}\right)$-crystal $B^{n, l}$ to $\nu b$ in the $U_{q}^{\prime}\left(A_{n-1}^{(1)}\right)$-crystal $B_{A}^{2, l}$. The operator $\nu$ will change $D_{n}^{(1)}$ to $A_{n-1}^{(1)}$ as a set,

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## List of Publications

## List of Peer-Reviewed Publications

[1] Mahathir bin Mohamad, Scattering Rules in Soliton Cellular Automata Associated with $U_{q}\left(D_{n}^{(1)}\right)$-Crystal $B^{n, 1}$, Journal of Physics A: Mathematical and Theoretical 45, (2012) 22 pages.
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## Seminar/ Workshop

[1] "Infinite Analysis 11 - Frontier of Integrability -", Poster Presentation, 2011.

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## The Author

