

WAVELETS OPERATIONAL METHODS FOR FRACTIONAL DIFFERENTIAL
EQUATIONS AND SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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For my beloved Parents.



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ABSTRACT

In this thesis, new and effective operational methods based on polynomials and wavelets for the solutions of FDEs and systems of FDEs are developed. In particular we study one of the important polynomial that belongs to the Appell family of polynomials, namely, Genocchi polynomial. This polynomial has certain great advantages based on which an effective and simple operational matrix of derivative was first derived and applied together with collocation method to solve some singular second order differential equations of Emden-Fowler type, a class of generalized Pantograph equations and Delay differential systems. A new operational matrix of fractional order derivative and integration based on this polynomial was also developed and used together with collocation method to solve FDEs, systems of FDEs and fractional order delay differential equations. Error bound for some of the considered problems is also shown and proved. Further, a wavelet bases based on Genocchi polynomials is also constructed, its operational matrix of fractional order derivative is derived and used for the solutions of FDEs and systems of FDEs. A novel approach for obtaining operational matrices of fractional derivative based on Legendre and Chebyshev wavelets is developed, where, the wavelets are first transformed into corresponding shifted polynomials and the transformation matrices are formed and used together with the polynomials operational matrices of fractional derivatives to obtain the wavelets operational matrix. These new operational matrices are used together with spectral Tau and collocation methods to solve FDEs and systems of FDEs.

ABSTRAK

Dalam tesis ini, kaedah operasi baru dan berkesan berdasarkan polinomial dan wavelet untuk penyelesaian persamaan pembezaan pecahan (PPP) dan sistem PPP dibangunkan. Khususnya, kami mengkaji satu polinomial penting yang tergolong dalam keluarga Appell, iaitu Genocchi polinomial. Polinomial ini mempunyai kelebihan tertentu dimana matriks operasi yang berkesan dan mudah buat pertama kalinya diperolehi dan digunakan bersama-sama dengan kaedah kolokasi untuk menyelesaikan persamaan pembezaan terbitan kedua jenis Emden-Fowler, persamaan Pantograph umum dan sistem persamaan pembezaan tertunda. Satu matriks operasi baru pembezaan dan kamiran pecahan yang berdasarkan polinomial ini juga telah dibangunkan dan digunakan bersama-sama dengan kaedah kolokasi untuk menyelesaikan PPP, sistem PPP dan persamaan pembezaan tertun da terbitan pecahan. Ralat batas untuk beberapa masalah juga ditunjukkan dan dibuktikan. Tambahan pula, satu basis wavelet berdasarkan polinomial Genocchi juga dibina, matriks operasi pembezaan peringkat pecahan diperolehi dan digunakan untuk penyelesaian PPP dan sistem PPP. Pendekatan baru untuk mendapatkan matriks operasi pembezaan peringkat pecahan berdasarkan Legendre dan Chebyshev wavelet juga dibangunkan, di mana, wavelet digubal daripada polinomial anjakan yang sepadan dan matriks transformasi dibentuk dan digunakan bersama dengan polinomial matriks operasi pembezaan peringkat pecahan untuk mendapatkan wavelet matriks operasi. Matriks operasi baru ini digunakan bersama-sama dengan kaedah Tau dan kaedah kolokasi spektrum untuk menyelesaikan PPP dan sistem PPP.

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LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbf{H}^{(\alpha)}$	Legendre wavelets operational matrix of fractional derivative
Φ	wavelets- polynomials transformation matrix
$\Phi_H(x)$	Shifted Chebyshev vector
$\Phi_L(x)$	Shifted Legendre vector
$\Psi(x)$	wavelets vector
$\Psi_{m,n}$	Wavelets basis
\mathbf{C}	Approximation coefficient vector
$\mathbf{D}^{(\alpha)}$	Shifted Legendre operational matrix of fractional derivative
$\mathbf{F}^{(\alpha)}$	Chebyshev wavelets operational matrix of fractional derivative
$\mathbf{G}(x)$	Genocchi polynomial vector
${}^h\phi$	Haar scaling function
${}^rD^\alpha$	Riemann-Liouville fractional derivative operator on $[0, 1]$
${}^rD_a^\alpha$	Riemann-Liouville fractional derivative operator on $[a, b]$
${}^c\mathbf{D}^{(\alpha)}$	Shifted Chebyshev operational matrix of fractional derivative
$B(x)$	Collection of best approximations
B_n	Bernoulli Number
D^α	Caputo fractional derivative operator on $[0, 1]$
D_a^α	Caputo fractional derivative operator on $[a, 1]$
G_n	Genocchi Number
$G_n(x)$	Genocchi polynomials of degree n
$H^{(\alpha)}$	Genocchi wavelets operational matrix of fractional derivative
I^α	Riemann-Liouville fractional integral operator on $[0, 1]$
I_a^α	Riemann-Liouville fractional integral operator on $[a, b]$
$L_i(x)$	Legendre polynomial
$P_i(x)$	Shifted Legendre polynomial

$S_n(x)$	Shifted Genocchi polynomials
$T_{H,i}(x)$	Shifted Chebyshev polynomial
$T_i(x)$	Chebyshev polynomial
U	Unit open ball (open ball centered at origin with radius 1)
M	Genocchi operational matrix of derivative



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CHAPTER 1

INTRODUCTION

1.1 Background

The concept of fractional calculus was first mentioned in a letter exchange traced back between Leibniz and L'Hopital [106, 147]. Leibniz introduced the notation $\frac{d^n y}{dx^n}$ (still used today) for n^{th} order derivative with the assumption that $n \in \mathbf{N}$, and reported this to L'Hopital. In his letter L'Hopital posed the question to Leibniz "...What would be the result if $n = \frac{1}{2}$?", Leibniz in his reply, dated 30th September 1695, writes "...this is an apparent paradox from which, one day, useful consequences will be drawn. Since there are little paradoxes without usefulness. ..." [106, 147]. S. F. de Lacroix was the first to introduced the fractional derivatives in published text in the year 1819. Subsequent contributions to fractional calculus were made by many great mathematicians of the time. Excellent summary of key milestones in the history of fractional calculus can be found in [109, 128]. Moreover, in a survey report [109], J. T. Machado, V. Kiryakova and F. Mainardi have comprehensively listed the major documents and key events in this area of mathematics since 1974 up to April 2010. Fractional calculus has long and rich history, but due to lack of suitable physical and geometrical interpretations, it remained unfamiliar to applied scientists up to recent years and was considered mathematical curiosities, not useful for solving problems arising from applied sciences. Several attempts have been made to provide physical and geometric interpretations for fractional operators. However, these interpretations are limited to only a small collection of selected applications of fractional derivatives

and integrals in the context of hereditary effects and self-similarity. Podlubny in [135] proposed a convincing physical and geometric interpretation of fractional derivatives and integrals. The authors in [84] interpret the geometrical meaning of the fractional order derivatives of any function.

There are several competing definitions of fractional derivatives and integrals. Some of them include, the Riemann-Liouville, the Caputo, the Hadamard, the Marchaud, the Granwald-Letnikov, the Erdelyi-Kober and the Riesz-Feller fractional derivatives and integrals. In general, these definitions are not equivalent except for some special cases. The most frequently used definition of fractional derivative and integral is due to B. Riemann and J. Liouville [134], commonly known as the Riemann-Liouville fractional derivative (integral). But in some situations, this approach is not useful due to lack of physical interpretation of initial and boundary conditions involving fractional derivatives, and also the Riemann-Liouville approach may yield derivative of a constant different from zero. A useful alternative to Riemann-Liouville derivative is the Caputo fractional derivative, introduced by Caputo in [28] and adopted by Caputo and Mainardi in the context of the theory of viscoelasticity [27].

For almost three centuries fractional calculus had been treated as an interesting, but abstract, mathematical concept. It had significantly been developed within pure mathematics. However the applications of the fractional calculus just emerged in last few decades in several diverse areas of sciences, such as physics, bio-sciences, chemistry and engineering. It is realized widely that in many situations fractional derivative based models are much better than integer order models. Being nonlocal in nature, the fractional derivatives provide an excellent tool for the understanding of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer order derivatives. A new application field for fractional calculus is psychological and life sciences, to characterize the time variation of emotions of people [7, 159]. In addition to the above mentioned applications, there are several applications of fractional calculus within different fields of mathematics itself. For example, the fractional operators are useful for the analytic investigation of various special functions [90, 91].

There are several collections of articles such as [74, 150], which exhibit wide variety of applications of fractional calculus and present many of the key developments of the theory.

The mathematical modelling and simulation of systems and processes, based on the description of their properties in terms of fractional derivatives, naturally lead to the formation of systems of differential equations of fractional order and to the necessity of solving such systems. However, effective general methods for solving them cannot be found even in the most useful works on fractional derivatives and integrals. Recently, orthogonal wavelets bases are becoming more popular for numerical solutions of differential equations due to their excellent properties such as ability to detect singularities, orthogonality, flexibility to represent a function at different levels of resolution and compact support. In recent years, there has been a growing interest in developing wavelet based numerical algorithms for solution of fractional differential equations. Wavelets have been successfully applied for the solutions of ordinary and partial differential equations, integral equations, and integro-differential equations of arbitrary order. Therefore, the main focus of this present research is the application of different wavelets techniques as well as polynomials techniques for solving systems of fractional differential equations.

1.1.1 Fractional Calculus

In the following, we study the most commonly used definitions of fractional integration and differentiation together with their important properties. We begin with the Riemann-Liouville fractional integration.

1.1.1.1 The Riemann-Liouville Fractional Integration

The common approach to define a fractional order integration is by the use of the well known Cauchy's integral formula for n -fold integral, i.e

$$I_a^\alpha f(x) = \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} f(x_0) dx_0 \cdots dx_{n-1} dx_n = \frac{1}{(n-1)!} \int_a^x (x-s)^{n-1} f(s) ds \quad (1.1)$$

where $f \in L^2[a, b]$, $a, b \in \mathbb{R}$. With the help of the generalization of factorial function to Gamma function one can replace n in (1.1) with an arbitrary real number α provided that the integral on right side converges. Thus, it is natural to define the fractional integral as follows:

Definition 1.1.1 [134] Let $f \in L^1[a, b]$, $\alpha \in \mathbb{R}^+$, the Riemann-Liouville fractional integral operator of order α is defined as

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \tau)^{\alpha-1} f(\tau) d\tau, \quad \forall x \in [a, b] \quad (1.2)$$

where $\Gamma(\cdot)$ is the well known Gamma function, when $\alpha = 0$ then we have, $I_a^\alpha = I$ identity operator.

We should also note that for arbitrary lower limit (1.2) is the Riemann version of fractional integral and for infinite lower limit, i.e., for $a = -\infty$, (1.2) is the Liouville version of fractional integral. The case when $a = 0$, i.e. I_0^α is called the Riemann-Liouville fractional integral. On the other hand, if we keep lower limit arbitrary, take upper limit to ∞ and replace kernel in (1.2) with $(s - x)^{\alpha-1}$ then the resulting integral operator, for a reasonable class of functions, is called the Weyl fractional integral of order α [83] and is usually denoted by ${}_x W_\infty^\alpha$.

Lemma 1.1.2 [134] If $\alpha \geq 0$, $\beta > -1$, then the Riemann-Liouville fractional integral of the function $(x - a)^\beta$ is given by

$$(I_a^\alpha (x - a)^\beta)(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} (t - a)^{\beta + \alpha}.$$

Before we discuss about the important property known as semi group property for the Riemann-Liouville fractional integral, we need the following theorem.

Theorem 1.1.3 [46] Let $\alpha \geq 1$ and $f \in L^1[a, b]$. Then, $I_a^\alpha f \in C[a, b]$.

Lemma 1.1.4 [89] Let $\alpha, \beta \in \mathbb{R}^+ \cup \{0\}$ and $f \in L^1[a, b]$. Then

$$I_a^\alpha I_a^\beta f(x) = I_a^{\alpha+\beta} f(x) = I_a^\beta I_a^\alpha f(x) \quad (1.3)$$

holds almost everywhere on $[a, b]$. Further, if $f \in C[a, b]$ or $\alpha + \beta \geq 1$, then (1.3) is identically true $\forall x \in [a, b]$.

1.1.1.2 The Riemann-Liouville Fractional Derivatives

With the concept of fractional integrals in mind, the notion of fractional order derivatives and some of their important basic properties is developed as follows.

Let D^n denotes the n^{th} order differential operator with $D^1 = D$. Then the fundamental theorem of integer order calculus becomes;

$$DI_a f = f \quad (1.4)$$

n -fold application of (1.4) yields the following relation

$$D^n I_a^n f = f, \quad n \in \mathbb{N}. \quad (1.5)$$

Replacing n in (1.5) by $m - n$ with $n < m$, $m \in \mathbb{N}$, we have

$$D^{m-n} I_a^{m-n} f = f. \quad (1.6)$$

Taking n^{th} derivative of both sides of (1.6), we have

$$D^n f = D^n D^{m-n} I_a^{m-n} f = D^m I_a^{m-n} f. \quad (1.7)$$

This relation is still valid and meaningful, for some reasonable class of functions, if n is replaced by $\alpha \in \mathbb{R}$ provided that $m \geq \lceil \alpha \rceil$. Using the semigroup property of fractional integrals together with the index law of classical derivative D and the fact that D is inverse of I , we have

$$D^\alpha f = D^m I_a^{m-\alpha} f.$$

It is worth noting that the operator defined in this way depends on the choice of lower limit a of fractional integral operator involved. Thus, fractional order derivative of a function is defined as follows.

Definition 1.1.5 [89] Let $\alpha \in \mathbb{R}^+$, and m an arbitrary integer such that $m > \alpha$ and $f \in C^m[a, b]$. The Riemann-Liouville fractional derivative of order α denoted by ${}^r D^\alpha$ is defined by

$${}^r D^\alpha f = D_a^m I_a^{m-\alpha} f. \quad (1.8)$$

Without loss of generality one can consider the narrow condition $m = \lceil \alpha \rceil$ or $m - 1 \leq \alpha < m$. Also in view of (1.7), the operator ${}^r D_a^\alpha$ coincides with classical n^{th} order derivative operator when α is replaced with a positive integer.

Lemma 1.1.6 [89] *Let $\alpha \geq 0$, $\beta > -1$, the Riemann-Liouville fractional derivative of a function $(x - a)^\beta$ is given by*

$$({}^r D_a^\alpha (x - a)^\beta)(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} (t - a)^{\beta - \alpha}.$$

Note: when $\beta = \alpha - i$, with $i = 1, 2, \dots, \lceil \alpha \rceil + 1$, we have ${}^r D_a^\alpha (x - a)^{\alpha - i} = 0$.

In the following, the composition relation between Riemann-Liouville fractional derivatives and integral is shown.

Lemma 1.1.7 [134] *If $\alpha, \beta \in \mathbb{R}^+$, $\alpha > \beta$ and $f \in L^1[a, b]$, then ${}^r D_a^\beta I_a^\alpha f = I_a^{\alpha - \beta} f$ holds almost everywhere on $[a, b]$.*

Proof. Using Definition 1.1.5 and semigroup property of fractional integrals, we have

$${}^r D_a^\beta I_a^\alpha f = {}^r D_a^{\lceil \beta \rceil} I_a^{\lceil \beta \rceil - \beta} I_a^\alpha f = I_a^{\alpha - \beta} f.$$

□

When $\alpha = \beta$ it immediately follows from above Lemma that the Riemann-Liouville fractional derivative is left inverse to fractional integral operator. For some restricted class of functions the Riemann-Liouville fractional derivative is also right inverse of fractional integral. The following Lemma is of great importance.

Lemma 1.1.8 [134] *Suppose that $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, then,*

$${}^r D_a^\alpha I_a^\alpha f(x) = f(x) \tag{1.9}$$

$$I_a^\alpha ({}^r D_a^\alpha f(x)) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{x^i}{i!}. \tag{1.10}$$

1.1.1.3 The Caputo Fractional Derivative

The Riemann-Liouville fractional differential operators have played a significant role in the development of the theory of differentiation and integration of arbitrary order. However, there are certain disadvantages of using the Riemann-Liouville fractional derivatives for modeling the real world phenomena. In Lemma 1.1.6 when $\beta = 0$ and $f(t) = C(t - a)^\beta$, one can easily see that the fractional derivative of constant, $({}^r D_a^\alpha(C))(t) = \frac{C(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$ is a function of t and is never zero except for $a = -\infty$. But for most of the physical applications the lower limit is required to be a finite number. It is also noted that the initial value problems for fractional differential equations with the Riemann-Liouville approach leads to the initial conditions involving fractional derivative at lower limit. Mathematically, such problems can successfully be solved. Being familiar with interpretation of real world problems with classical derivatives, it is commonly known up till now we do not have any known physical interpretation of initial conditions involving fractional derivatives. Applied problems, modeled using fractional operators, require an approach to fractional derivatives which can utilize physically meaningful initial conditions involving classical derivatives. To cope with these situations, Caputo in [28] introduced another definition of fractional derivative and later in 1969 Caputo and Mainardi used it in the framework of viscoelasticity theory [27]. In what follows, the formal definition of the Caputo derivative and its relations with the Riemann-Liouville fractional integral and derivative is given.

Definition 1.1.9 [89] *Let $\alpha \in \mathbb{R}^+$ and $f \in C^m[a, b]$, $m = \lceil \alpha \rceil$. Then the Caputo fractional derivative of order α is defined by*

$$D_a^\alpha f(x) = I^{m-\alpha} D^m f(x) \quad (1.11)$$

for $m = \alpha$ the equation yields $D_a^\alpha f(x) = D^m f(x)$. Thus for integer values of α , the Caputo fractional derivative becomes the conventional derivative.

It is obvious from definition of Caputo derivative that $D_a^\alpha(C) = 0$, where C is constant. The definitions of fractional derivative, both in the sense of Riemann-Liouville and Caputo, utilize the definition of Riemann-Liouville fractional integral but the order of fractional integration with integer differentiation is interchanged. Also, note that

both in the definition of the Riemann-Liouville fractional derivative and the Caputo fractional derivative it is required that $m = \lceil \alpha \rceil$. This condition is not strict in the case of the Riemann-Liouville definition of fractional derivative. one may chose any integer m such that $m \geq \alpha$. However in the case of the Caputo fractional derivative, the condition $m \geq \lceil \alpha \rceil$ may not be used. Another difference between Riemann-Liouville and caputo fractional derivatives is that the Riemann-Liouville fractional derivative exists for a class of integrable function while the existence of the Caputo fractional derivative requires the integrability of m times differentiable functions. This can be seen from following Lemma.

Lemma 1.1.10 [46] *If $\alpha \geq 0$ and $f(x) = (x - a)^\beta$, $m = \lceil \alpha \rceil$, where $\lceil \alpha \rceil$ denote the smallest integer greater than or equal to α and $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α , then*

$$D_a^\alpha f(x) = \begin{cases} 0, & \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta < \lceil \alpha \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (x-a)^{\beta-\alpha}, & \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta \geq \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \alpha \rfloor. \end{cases}$$

Similar to the integer order differentiation, the Caputo factional differential operator is a linear operator, Since,

$$D^\alpha(\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x) \quad (1.12)$$

where λ and μ are constants. We end this subsection by recalling the definition of fractional derivative due to Grunwald-Letnikov

1.1.1.4 Grunwald-Letnikov Fractional Derivative

The Grunwald-Letnikov fractional derivative is defined as [134]

$${}_a D_x^\alpha f(x) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{\substack{r=0 \\ nh=x}}^n (-1)^r \binom{\alpha}{r} f(x - rh). \quad (1.13)$$

1.1.2 Wavelets

The origin of wavelets goes back to the beginning of 20th century, when Alfred Haar in 1910 constructed an orthonormal system of functions on the unit interval $[0, 1]$ which led to the development of a set of rectangular basis functions [62]. Historically, the concept of wavelets was formally introduced at the beginning of eighties by J. Morlet, a French geophysical engineer, as a family of functions constructed by translation and dilation of single function, called the mother wavelet [62, 99, 143]. The reason behinds the discovery of wavelets is that Fourier series represents frequency of a signal, but it does not model its localized features appropriately. This is because the building blocks of Fourier series, the sine and cosine functions, are periodic waves which continue forever.

The wavelet theory have drawn great deal of attention from scientists working in various disciplines because of its comprehensive mathematical power and wide range applications in science and engineering. Particularly, wavelets are very useful in signal processing, image processing, edge extraction, computer graphics, approximation theory, biomedical engineering, differential equations, numerical analysis, etc. Wavelets are special kind of functions which exhibit oscillatory behavior for a short period of time and then become zero. Wavelets are constructed from dilation and translation of single function $\psi(t)$, called mother wavelet and thus generating a two parameter family of functions $\psi_{a,b}(t)$. $\psi_{a,b}(t)$ is defined as follows.

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0$$

where a is dilation parameter and b is translation parameter. If $|a| < 1$, then $\psi_{a,b}$ is compressed form of mother wavelet and corresponds to higher frequencies. On the other hand, for $|a| > 1$ the wavelet $\psi_{a,b}$ corresponds to lower frequencies. More precisely, the following is the definition of wavelets:

Definition 1.1.11 [41] *A function $\psi \in L^2(\mathbb{R})$ is admissible as a wavelet if and only if*

$$A_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

where $\hat{\psi}$ is the Fourier transform of ψ .

The admissibility condition requires that A_ψ is finite, this means $\hat{\psi}(0) = 0$, i.e the mean value of ψ should vanish; $\int_{-\infty}^{\infty} \psi(s) ds = 0$.

The continuous wavelets are not useful for many practical purposes. In particular they do not form basis. For that reason, wavelets are discretized by fixing the positive constants $a_0 > 1$, $b_0 > 0$ and setting $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$ where $n, k \in \mathbb{N}$. Thus, the following family of discrete wavelets are defined as

$$\psi_{k,n}(x) = |a_0|^{\frac{1}{2}} \psi(a_0^k x - nb_0).$$

Usually a_0 is chosen to be 2 and $b = 1$. Ingrid Daubechies gave solid foundations for wavelet theory. In [40], she provided major break through by constructing a system of orthonormal wavelets with compact support. The Haar wavelet is the simplest example of orthogonal wavelets compactly supported on the interval $[0, 1]$ and was constructed by Haar in 1910 in his Ph.D. dissertation [62].

1.1.2.1 Haar Scaling Functions

In discrete wavelet transform we consider two sets of functions, scaling functions and wavelet functions. The Haar scaling function ${}^h\phi$ is defined on interval $[0, 1]$ as

$${}^h\phi(x) = \chi_{[0,1)}(x) = \begin{cases} 1, & x \in [0, 1) \\ 0, & \text{elsewhere} \end{cases} \quad (1.14)$$

which is a characteristics function of the interval $[0, 1)$. The translates of Haar scaling function $\{{}^h\phi(x - k)\}_{k \in \mathbb{Z}}$ form an orthonormal set of functions. That is

$$\int_{\mathbb{R}} {}^h\phi(x - m) {}^h\phi(x - n) dx = \delta_{m,n}.$$

The subspace of $L^2(\mathbb{R})$ spanned by translates of the Haar scaling functions is denoted by V_0 . Scaling translates of $\phi(x)$ by 2^i , we get the functions ${}^h\phi_{i,k} = 2^{\frac{i}{2}} ({}^h\phi(2^i x - k))$ supported on the dyadic sub-intervals $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$, $j, k \in \mathbb{Z}$. For fixed i , the functions $\{{}^h\phi_{i,k}\}$ are orthonormal among themselves and the space spanned by $\{{}^h\phi_{i,k}\}$

is denoted by V_i . The Haar scaling function ${}^h\phi$ satisfies the dilation equation

$${}^h\phi(x) = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k \phi(2x - k) \quad (1.15)$$

where c_k is given by

$$c_k = \sqrt{2} \int_{-\infty}^{\infty} ({}^h\phi(s)) ({}^h\phi(2s - k)) ds \quad (1.16)$$

evaluating (1.16) we have

$$c_0 = \frac{1}{\sqrt{2}}, \quad c_1 = \frac{1}{\sqrt{2}}$$

and $c_k = 0, \forall k > 1$. Therefore, the dilation equation becomes

$${}^h\phi(x) = {}^h\phi(2x) + {}^h\phi(2x - 1). \quad (1.17)$$

The spaces spanned by the scaling functions, define a multiresolution representation in $L^2(\mathbb{R})$. The idea of multiresolution is to express functions in $L^2(\mathbb{R})$ as limit of successive approximations. These successive approximations use different levels of resolutions. In the following subsection, we define multiresolution analysis.

1.1.2.2 Multiresolution Analysis (MRA)

As a more general framework we explain Mallat's Multiresolution Analysis (MRA). The MRA is a tool for a constructive description of different wavelet bases.

Definition 1.1.12 A multiresolution analysis is a sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$, that satisfy the following conditions;

- (i). The sequence $\{V_j\}_{j \in \mathbb{Z}}$ is nested, i.e $\cdots V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_n \subset V_{n+1} \subset \cdots$
- (ii). $\bigcup_{i \in \mathbb{Z}} V_i$ is dense in $L^2(\mathbb{R})$, i.e $\overline{\bigcup_{i \in \mathbb{Z}} V_i} = L^2(\mathbb{R})$
- (iii). $\bigcap_{i \in \mathbb{Z}} V_i = \{0\}$
- (iv). $f(x) \in V_i$ if and only if $f(2x) \in V_{i+1}$
- (v). There exists a function $\phi \in V_0$ such that $\{{}^h\phi_{0,k} = {}^h\phi(x - k), k \in \mathbb{Z}\}$ is a Riez basis for V_0 , that is, for every $f \in V_0$, there exists a unique sequence $\{c_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$

such that $f(x) = \sum_{k \in \mathbb{Z}} c_k^h \phi(x - k)$ with convergence in $L^2(\mathbb{R})$ and there exist two positive constants M, N independent of $f \in V_0$ such that

$$M \sum_{k \in \mathbb{Z}} |c_k|^2 \leq \|f\|^2 \leq N \sum_{k \in \mathbb{Z}} |c_k|^2$$

where $0 < M < N < \infty$.

The subspaces $\{V_i\}_{i \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ spanned by the sets of functions $\phi_{i,k}$ is a MRA. Thus the Haar scaling function generates the MRA.

1.1.2.3 Haar Wavelets

Based on the relation $V_j \subset V_{j+1}$ we are interested in the decomposition of V_{j+1} as an orthonormal sum of V_j and its orthogonal complement. For $j = 0$, the space V_0 is spanned by the integer translates of scaling function $^h\phi$. This observation motivates for the construction of a function $\psi(x)$ whose translate form the basis for the orthogonal complement V_0^\perp of V_0 . The function ψ should be member of V_1 and orthogonal to V_0 . The simplest ψ that fulfills these requirements is $\psi(x) = \chi_{[0, \frac{1}{2}]} - \chi_{[\frac{1}{2}, 1]}$ and is referred as Haar wavelet function. The translates of ψ i.e. $\{\psi(x - k)\}_{k \in \mathbb{Z}}$ form an orthonormal set. Furthermore it is easy to see that the system $\{\psi(x - k)\}_{k \in \mathbb{Z}}$ forms basis for the space V_0^\perp . For each pair of $j, k \in \mathbb{Z}$, the dilated translates of ψ are given as

$$\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k). \quad (1.18)$$

The functions $\psi_{j,k}(x)$ are compactly supported on the dyadic intervals $I_{j,k}, j, k \in \mathbb{Z}$. For fixed $j \in \mathbb{Z}$, we define V_j^\perp as the orthogonal complement of $V_j \subset V_{j+1}$. The system $\{\psi(x - k)\}_{k \in \mathbb{Z}}$, forms an orthonormal basis of the complementary space V_j^\perp .

1.2 Problem Statements

FDEs have attracted considerable interests because of their ability to formulate complex phenomena in the fields of physics, chemistry, engineering, aerodynamics, electrodynamic of complex medium, polymer rheology and etc, [6, 89, 134]. Due to

their extensive applications, considerable interest in developing numerical schemes for their solutions is becoming very attractive research area, as such, several methods are established. Adomian decomposition method [140], variational iteration method [177], homotopy perturbation method [126] and predictor-corrector method [47] are some of the famous methods. The idea of approximating the solutions of FDEs by family of basis functions have been widely used and it turns out to yield very good results when compared with some other existing methods. The most commonly used functions are Block pulse functions [98, 179], Legendre polynomials [149], Chebyshev polynomials [50], Laguerre polynomials [2, 22, 23] and so on. Furthermore, wavelets are just another basis sets that offers considerable advantages over alternative basis sets and allow us to tackle problems not accessible with conventional numerical methods, these main advantages are discussed in [60]. Legendre wavelets [77, 144, 163], chebyshev wavelets [182] and Bernoulli wavelets [85] are few examples of the wavelets methods applied for solving FDEs. However, it should be noted that much of the researches published to date, concerning exact and numerical solutions of FDEs are devoted to the initial value problems or boundary value problem for single FDEs. The systems of fractional differential equations has received less attention in this regard, despite the fact that most of the real world processes modeled using fractional calculus results in systems of fractional differential equations. Therefore, it is the purpose of this research to propose an effective and simple operational method for the solution of systems of fractional differential equations.

1.3 Objectives

The main objectives of this research is to investigate the applications of wavelets operational algorithms on systems of FDEs in particular, the specific objectives are:

- (i) To develop new and efficient polynomials and wavelets operational algorithms (i.e Genocchi polynomials) for the solution of FDEs and systems of FDEs.
- (ii) To derive more simpler method of obtaining operational matrix of fractional derivative for some of the existing wavelets (Legendre and Chebyshev wavelets)

for the solution of FDEs and systems of FDEs.

- (iii) To Investigate the error bounds of the approximate solutions using these polynomials and wavelets operational method for the FDEs and systems of FDEs.

1.4 Scope of Study

In this research, we will focus on polynomials and wavelets operational matrix of fractional order derivatives of Genocchi, Legendre and Chebyshev polynomials and wavelets bases. We will show a new simple algorithms based on these bases, its accuracy and effectiveness when solving systems of fractional differential equations through some error analysis. Many numerical examples are considered to clearly justify the accuracy of the new algorithms in comparison with other methods.

1.5 Main Contributions

This present research provides a new algorithms based on the Genocchi polynomials and wavelets, Legendre wavelets and Chebyshev wavelets for the solution of FDEs and systems of FDEs. Therefore, the novelty of this work goes to the new developed wavelets operational method based on Genocchi polynomials and the modification made on the Legendre and Chebyshev wavelets operational algorithms, in which new way to obtain the wavelets operational matrices of fractional derivatives is developed.

1.6 Thesis Outline

This thesis consists of six chapters. First chapter is the brief introduction and background of the research as well as the objectives of the research. Chapter two highlights the information from the previous works relevant to this research in order to have a deeper understanding of the scope in this thesis. In Chapter three, we introduce the operational method for solving FDEs where we will focus on the methods followed by previous literature to obtain operational matrices of fractional

order derivative and its application in solving FDEs and systems of FDEs. In Chapter four, The Genocchi polynomials and Genocchi wavelets are introduced together with their operational matrices which is applied through collocation method to solve several FDEs and systems of FDEs. While in chapter five, new way to obtain Legendre and Chebyshev wavelets operational matrices of fractional derivative through wavelets-polynomial transformation is shown and applied together with collocation method to obtain solutions of FDEs and systems of FDEs. Conclusion and recommendations are given in Chapter six.



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CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, the current state of knowledge in the analytical and numerical approaches for solving FDEs and systems of FDEs is explored. The mathematical modelling of numerous processes in various areas of science and engineering using fractional derivatives naturally leads to the formation of FDEs. Although the fractional calculus has a long history and has been applied in various fields of study, the interest in the study of FDEs and their applications has attracted the attention of many researchers and scientific societies beginning only in the last three decades. However, the exact solutions of most of the FDEs can not be found easily, thus analytical and numerical methods must be used.

2.2 Analytical Methods for Solving FDEs

Various analytical methods for solving FDEs problems are employed by many researchers, the most used methods include the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM). For example, in [4, 122, 141] ADM is used to solve fractional diffusion equations and linear and nonlinear fractional differential equations, respectively. In [76] the authors presented an enhanced HPM to obtain an approximate solution of FDEs, and Abdulaziz et al. [5]

extended the application of HPM to systems of FDEs. However, the convergence region of the corresponding results is rather small. S.J. Liao in [100, 101, 102, 103, 104] used the basic ideas of the homotopy in topology to come up with a general analytic method for nonlinear problems, known as, Homotopy Analysis Method (HAM). This method has been successfully applied to solve many types of nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [65], the Kdv equations [1], higher dimensional initial boundary value problems of variable coefficients [79], linear and nonlinear fractional partial differential equations are solved in [3]. Using HAM, the solutions of some Schrodinger equations are exactly obtained in the form of convergent Taylor series [9] and finance problems [188]. The HAM contains a certain auxiliary parameter h which provides a simple way to adjust and control the convergence region and rate of convergence of the series solution. Another powerful method which can also give explicit form for the solution is the variational iteration method (VIM). It was proposed by He [69, 70], and other researchers have applied VIM to solve various problems in [66, 67, 68]. Also, Song et al. [160] used VIM to obtain approximate solution of the fractional Sharma-Tasso-Olever equations. Yulita Molliq et al. [118, 119] solved fractional Zhakanov-Kuznetsov and fractional heat and wavelike equations using VIM to obtain the approximate solution have shown the accuracy and efficiency of VIM. Nevertheless, VIM is only valid for short time interval for solving the fractional system. In [183], a modification of VIM to overcome this weakness is proposed.

2.3 Numerical Methods for Solving FDEs

Several methods for the approximate solutions to classical differential equations are extended to solve differential equations of fractional order numerically. These methods include, generalized differential transform method [127], finite difference method [96], fractional linear multi-step method [107], extrapolation method [48] and predictor-corrector method [47].

2.3.1 Polynomial and Orthogonal Functions Operational Methods

Recently, the idea of approximating the solutions of FDEs by family of orthogonal basis functions have been widely used, Saadatmandi and Dehghan [149] introduced shifted Legendre operational matrix for fractional derivatives and applied it with spectral methods for numerical solution of multi-term linear and nonlinear FDEs. Doha et al. [50] derived a new formula expressing explicitly any fractional order derivatives of shifted Chebyshev polynomials of any degree in terms of shifted Chebyshev polynomials themselves, and used it together with tau and collocation spectral methods to find an approximate solutions for multi-term linear and nonlinear FDEs, Atabakzadeh et al. [14] used the operational matrix of Caputo fractional-order derivatives for Chebyshev polynomials to solve a system of FDEs, Bhrawy et al. [21] used a quadrature shifted Legendre tau method for treating multi-order linear FDEs with variable coefficients, Doha et al. [50] introduced shifted Chebyshev operational matrix and applied it with spectral methods for solving multi-term linear and nonlinear FDEs subject to initial and boundary conditions, Doha et al. [51] also introduce the shifted Jacobi operational matrix of fractional derivative which is based on Jacobi tau method for solving numerically linear multi-term FDEs with initial or boundary conditions. They also introduce a suitable way to approximate the nonlinear multi-term fractional initial or boundary value problems on the interval $[0, L]$, by spectral shifted Jacobi collocation method based on Jacobi operational matrix. In [2] a new approach implementing Laguerre operational matrix in combination with the Laguerre collocation technique is introduced for solving nonlinear multi-term FDEs, where as in [23] the authors derived the operational matrix of Riemann-Liouville fractional integration of Modified generalized Laguerre polynomials (MGLP) and applied it for approximating the linear FDEs on the half-line. Furthermore, the operational matrix of Caputo fractional derivatives of MGLP is used in combination with the spectral tau scheme for approximating linear FDEs, and with the pseudo-spectral scheme for approximating nonlinear FDEs on the half-line [22] in which it was stated that the Laguerre operational matrix and the generalized Laguerre operational matrix [18] can be obtained as special cases of MGLP. Bernstein's approximation is used in [156, 157] to find the stable solution of the

problem of Abel inversion, while in [132] Bernstein operational matrix of fractional order integration is developed and is applied for solving linear and non linear FDEs. Bernoulli and Euler polynomials are also very important functions when it comes to an arbitrary function approximation, they are not based on orthogonal functions but they possess both operational matrix of derivative and integration. In [137] the properties of Bernoulli polynomials are used to construct the operational matrices of integration together with the derivative and product to approach differential equations. Tohidi et al [162] they used collocation method based on Bernoulli operational matrix of derivatives for the numerical solution of the generalized pantograph equation with variable coefficients subject to initial conditions. New operational matrix based on hybrid Bernoulli and Block-Pulse functions has been developed to approximate the solution of system of linear Volterra integral equations in [64]. Euler polynomials operational matrix of derivative and integration also played an important role in solving FDEs and fractional integral equations for instance some works that used Euler polynomials as basis for numerically solving integral, differential, and integro-differential equations. In [113, 114], operational matrices for integration and differentiation of Euler polynomials are introduced. A new operational matrix of differentiation for these polynomials are also used via collocation method to convert nonlinear fractional Volterra integro-differential equations to the associated systems of algebraic equations in [115]. In [112] the authors used the Euler polynomials to construct the approximate solutions of the second-order linear hyperbolic partial differential equations with two variables and constant coefficients in which a formula expressing explicitly the Euler expansion coefficients of a function with one or two variables is proved. The authors also show explicit formula which expresses the two dimensional Euler operational matrix of differentiation Application of these formulae for reducing the problem to a system of linear algebraic equations with the unknown Euler coefficients, is then explained.

Another interesting method is to divide the interval into a collection of sub-intervals and construct a generally distinct approximating polynomial on each sub-interval. This is called B-spline function. We may note that the "B" in B-spline stands for basis. Spline functions are instances of a piecewise polynomial function associated with a partition of an arbitrary interval. Approximation by functions of this type is

called piecewise-polynomial approximation. The main applications of B-splines arise in computer-aided design, computer graphics, geometric modeling and many other different subjects [52]. With the help B-spline collocation techniques many authors suggested the operational matrix of fractional derivative to solve fractional differential equations as in [95] where the authors introduce a new operational method based on B-spline operational matrix of fractional derivative to solve multi-order fractional differential and partial differential equations by expanding the solution as linear B-spline functions with unknown coefficients. Also in [81] the authors generalize the operational matrix for fractional integration and multiplication to solve fractional Riccati problems. Operational matrices based on cubic B-spline scaling functions are also constructed and used in solving many kinds of FDEs. In the work of Xinxu [97] they constructed the cubic B-spline operational matrix of fractional derivative in the Caputo sense, and use it to solve fractional differential equation. In [94] cubic B-spline scaling functions operational matrix of derivative is used to reduced the solution of Fokker- Planck equations. Many more operational matrices of fractional derivatives and integration based on other polynomials for the solution of FDEs can be found in open literature.

2.3.2 Wavelets Basis Operational Methods

Wavelets are localized functions, which are the basis for $L^2(\mathbb{R})$. So that localized pulse problems can be easily approached and analyzed [29, 30, 31]. They are used in system analysis, optimal control, numerical analysis, signal analysis for wave form representation and segmentations, time-frequency analysis and fast algorithms for easy implementation [144]. However, wavelets are just another basis set which offers considerable advantages over alternative basis sets and allows us to attack problems not accessible with conventional numerical methods. Their main advantages are as [60] the basis set can be improved in a systematic way, different resolutions can be used in different regions of space, the coupling between different resolution levels is easy, there are few topological constraints for increased resolution regions, the Laplace operator is diagonally dominant in an appropriate wavelet basis, the matrix elements of the Laplace operator are very easy to calculate and the numerical effort

scales linearly with respect to the system size. Different wavelets operational matrices of fractional order integration and differentiation for solving FDEs have been developed by many researchers for example see [71, 99, 123, 143]. We discuss existing literature of some examples of the wavelets in the following sections.

2.3.2.1 Haar Wavelets Operational Methods

Haar wavelets are the mathematically simplest of all wavelet families and was constructed by Haar in 1909 in his Ph.D. dissertation. Historically, Chen and Hsiao [33], first proposed a Haar operational matrix for the integration of Haar function vectors and used it for solving differential equations. Since then numerous operational matrices based on different orthogonal functions emerge. Haar wavelets permit straight inclusion of the different types of boundary conditions in the numerical algorithms. Another good feature of these wavelets is the possibility to integrate them analytically in arbitrary times. However, the major drawback of these wavelets is their discontinuity, since the derivatives do not exist at the partition points, so the integration approach is preferred instead of the differentiation for calculation of the coefficients. Authors in [99, 139] have successfully applied the Haar wavelet operational matrix of fractional order integration to solve FDEs numerically, while [165] have obtained sufficient conditions for the existence and uniqueness of solutions for a class of fractional partial differential equations using Haar wavelet operational matrix of fractional order integration. In [154] a new Haar wavelet method based on operational matrices of fractional order integration to solve several types of fractional order differential equations numerically is presented, the method proposed through which the operational matrices of fractional order integration were formed does not require the use of the block pulse functions and inverse of Haar wavelet matrix. Furthermore, this procedure is shown to consume less CPU time as the major blocks of Haar wavelet operational matrix are calculated once and, are used in the subsequent computations repeatedly.

2.3.2.2 Chebyshev Wavelets Operational Methods

Chebyshev wavelets has been widely applied for solving different functional equations, In [71], the Chebyshev wavelets operational matrix of fractional-order integration is derived and employed to reduce the multi-order FDEs to systems of nonlinear algebraic equations, Saeed in [158] used Chebyshev wavelets for numerical solution of Abel's integral equation, a computational method based on the second kind Chebyshev wavelet for solving a class of nonlinear Fredholm integro-differential equations of fractional order is presented in [187], An extension of Chebyshev wavelets method for solving nonlinear systems of Volterra integral equations is explored in [24], the authors in [123] used Chebyshev wavelets expansions together with operational matrix of derivative to solve ordinary differential equations in which, at least, one of the coefficient functions or solution function is not analytic. For more works on Chebyshev wavelets see [15, 57, 72, 166] and references therein.

2.3.2.3 Legendre Wavelets Operational Methods

Legendre wavelets method is also thoroughly applied for solving differential equations. In [82], a framework to obtain approximate numerical solutions of FDEs using Legendre wavelet approximations is developed, they used the properties of Legendre wavelets to reduce the fractional ordinary differential equations to the solution of algebraic equations. In [110], multi-projection operators to solve the linear Fredholm integral equation of the second kind with Legendre wavelets is shown. An operational matrix of fractional order integration based on Legendre wavelets is derived and is utilized to reduce the FDEs to system of algebraic equations in [163]. In [142], a direct method for solving variational problems using Legendre wavelets is presented. The authors in [143] introduce a new method based on Legendre wavelets operational matrix of integration to solve nonlinear problems of the calculus of variations. The operational matrix of integration is used to evaluate the coefficients of Legendre wavelets in such a way that the necessary conditions for extremization are imposed. In [181], application of Legendre wavelets to the numerical solution of

Lane-Emden equations is discussed. The method first convert Lane-Emden equations to integral equations and then expand the solution by Legendre wavelets with unknown coefficients. A numerical method based upon Legendre wavelet approximations for solving nonlinear Volterra-Fredholm integral equations is presented in [180]. Operational matrix of derivative based on Legendre wavelets is constructed by using shifted Legendre polynomials in [116], application of this operational matrix for solving initial and boundary value problems are then described while in [111] Legendre wavelets operational matrix of fractional order integration is used for the solution of linear and nonlinear fractional integro-differential equations and the upper bound for the Legendre wavelets expansion is shown. Recently, the first paper by Yimin used Legendre wavelets operational methods to solve nonlinear system of FDEs [37]. Our contributions in this particularly focus on construction of a new algorithm for obtaining Legendre wavelets operational matrix of fractional order derivative and this operational methods are applied for the solution of fractional order Brusselator system [32].

2.3.2.4 Bernoulli Wavelets Operational Methods

Another important wavelets basis is Bernoulli wavelets, this wavelets are not based on orthogonal functions, but, they possess the operational matrices of integration and derivative [17, 85]. They have been widely applied for solving FDEs, for instance, in [85] an operational matrix of fractional order integration based on Bernoulli wavelets is derived and is utilized to reduce the initial and boundary value problems to system of algebraic equations. Balaji [17] derived Bernoulli wavelets operational matrix of derivative and apply it to solve Lane-Emden equations. In the proposed method the nonlinear derivatives of Lane-Emden equations are directly replaced by Bernoulli wavelet series using Bernoulli wavelet operational matrix of derivative. Further the non linearity of unknown function resolved by taking suitable collocation points. In this procedure, there is no necessity of conversion of the Lane-Emden equation into integral equations and no iterations is required to remove the non linearity in the Lane-Emden equation. This indeed provides the advantage of proposed method over other wavelet method in terms of less computational effort and time for getting good

approximate solution to the Lane-Emden type equations. Recently, P.K Sahu [151] developed Bernoulli wavelets method to solve nonlinear weakly singular Volterra integro-differential equations. More recently, P Rahimkhan in [138] introduced a Bernoulli wavelet operational matrix of fractional integration and proposed the application of the operational matrix for the solution of fractional order delay differential equations. He also proposed a new fractional function based on the Bernoulli wavelet to obtain a solution for systems of FDEs. The method entails expanding the considered approximate solution as the elements of the fractional-order Bernoulli wavelets, then the operational matrices of fractional order integration and derivative based on fractional order Bernoulli wavelets are derived. These operational matrix of fractional integration together with collocation method are utilized to reduce the initial value problems to system of algebraic equations.

2.4 Summary

This review of the solution techniques for FDEs serves as a base to begin to identify the importance and challenges in constructing new numerical approximation schemes for solving FDEs. One notes from the above discussion that numerical methods for the FDEs comprise a very new and fruitful research field. Although the majority of the previous research in this field has focused on single FDE problems, the systems of FDEs received less attention. Furthermore, some of the wavelets algorithms looks somehow complicated and used much CPU time in computations. Therefore, this motivates us to consider effective, simple and fast wavelets numerical algorithms for the FDEs and systems of FDEs.

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