SIR EPIDEMIC AND PREDATOR - PREY MODELS OF FRACTIONAL-ORDER

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This Masters Project Report is humbly dedicated to Almighty God, my wife and my children;

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ABSTRACT

Recently, many deterministic mathematical models such as ordinary differential equations have been extended to fractional models, which are transformed using fractional differential equations. It was believed that these fractional models are more realistic to represent the daily life phenomena. The main focus of this report is to extend the model of a predator-prey and the SIR epidemic models to fractional model. More specifically, the fractional predator-prey model which depend on the availability of a biotic resources was discussed. On the other hand, fractional SIR epidemic model with sub-optimal immunity, nonlinear incidence and saturated recovery rate was also discussed. The fractional ordinary differential equations were defined in the sense of the Caputo derivative. Stability analysis of the equilibrium points of the models for the fractional models were analyzed. Furthermore, the Hopf bifurcation analysis of each model was investigated . The result obtained showed that the model undergo Hopf bifurcation for some values. Throughout the project, the Adams-type predictor-corrector method to obtain the numerical solutions of the fractional models was applied. All computations were done by using mathematical software, Maple 18.

ABSTRAK

Kebelakangan ini, banyak model matematik yang terdiri dari persamaan pembezaan biasa telah dilanjutkan ke model pecahan, dimana menggunakan persamaan pembezaan pecahan. Adalah dipercayai bahawa model pecahan adalah lebih realistik untuk mewakili fenomena kehidupan seharian. Tumpuan utama laporan ini adalah untuk memperluaskan model pemangsa mangsa dan wabak SIR kepada model pecahan. Secara terperinci, tentang model pemangsa mangsa pecahan yang bergantung kepada kewujudan sumber biotik telah dincangkan. Selain daripada itu, model wabak SIR pecahan dengan imuniti sub-optimum, kadar tak lelurus dan kadar pemulihan tepu juga dibincangkan. Persamaan pembezaan pecahan ini didefinisikan menggunakan pembeza Caputo. Analisis kestabilan mengenai titik keseimbangan untuk model pecahan juga dianalisis. Tambahan pula, analisis bifurkasi Hopf bagi setiap model telah disiasat. Hasil yang diperoleh menunjukkan bahawa model ini mempunyai bifurkasi Hopf untuk beberapa nilai tertentu. Untuk keseluruhan projek, kaedah peramal-pembetul jenis Adams untuk memperoleh penyelesaian berangka bagi model pecahan telah digunakan. Kesemua pengiraan dijalankan dengan menggunakan perisian matematik, Maple 18.

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LIST OF SYMBOLS AND ABBREVIATIONS

Predator-Prey Model

${cD_t}^{\alpha}$ - The Caputo fractional derivative with order $\alpha \in (0, 1]$

$D(p)$ - The discriminant of a polynomial

X - Prey population

Y - Predator population

Z - The limited available of the biotic resources

r_1 - The logistical growth rate of the prey

r_2 - The logistical growth rate of the predator

pZ - The environmental carrying capacity for the prey population

qZ - The environmental carrying capacity for the predator population

c - The developmental rate growth of the biotic resources

dZX - The rate of uptake of the resources by the prey

eZY - The rate of uptake of the resources by the predator

t - Time

a, b, c, d, e - The constant values

λ - The eigenvalues at each steady-state

J - Jacobian matrix

Epidemic Model

${cD_t}^{\alpha}$ - The Caputo fractional derivative with order $\alpha \in (0, 1]$

$D(p)$ - The discriminant of a polynomial

I - Infected population

R - Recovered population

$N(t)$ - The total number of the population at a particular time, t

β - The disease transmission rate

A - The recruitment rate

μ - The natural death rate

v - The recovery rate of the infected population with and with no treatment

k - The rate where the recovered individual leaves the immunity

R_0 - The basic reproduction number

E^0 - Disease-free equilibrium

E^e - Endemic equilibrium

λ - The eigenvalues at each steady-state

t - Time

J - Jacobian matrix

CHAPTER 1

INTRODUCTION

1.1 Background of study

Biology is a natural science that is concerned with the study of living organisms, which is often studied in conjunction with other sciences including mathematics and engineering, and of course social sciences. On top of this, mathematical biology has been a triumphant mechanism in health issues (Lewis, 2008; Rostamy and Mottaghi, 2016). There are many reasons for this. Many attempts have been made to develop realistic mathematical models by several researchers in order to make prediction about the living organism's behaviour. Among such models are epidemics and predator-prey models, which are the focus in this project.

The study of epidemiology, concerning the transmission of diseases within a population, has been an active research among researchers in various fields. The SARS outbreak in 2003 and Ebola outbreak in 2014 had even speeding the research in this area. On top of that, many mathematical models of infectious diseases have been developed in order to study the dynamical process of epidemic. The models are able to integrate realistic aspects of disease spreading. Historically, a simple deterministic model was dated back by Kermack and McKendrick in 1927, which is the susceptible-infected- recovered (SIR) model. In this model, populations are divided into three states, which are susceptible, S, infected, I, and recovered (removed), R, respectively. Originally, it is assumed that susceptible individuals become infected with a rate of transmission which is proportional to the fraction of infected individuals in the overall population (fully mixed approximation) and infected individuals recover at a constant rate.

In this project direction, mathematicians and physicists are among the academic players who contribute to the knowledge of mathematical epidemiology for investigating the transmission dynamics of infectious diseases and the asymptotic behaviors of these epidemic models. They constantly working in modelling the epidemics outbreak. One of the main parts of epidemiological research is focus on the rate-based differential-equation models, i.e. compartmental models on completely mixing population. This epidemiology modelling has been used in planning, implementing and evaluating various prevention therapy and control programs. Before the era of research in complex networks, the theoretical approach to epidemic spreading is based on compartmental models in term of system of ordinary differential equations (ODEs). Two typical epidemiology models in the ODEs form, are the SIS (susceptible-infected-susceptible) model and the SIR (susceptible-infected- refractory) model. In the SIS model, an individual has two possible statuses: susceptible and infected. A susceptible individual may become infected once it contacts an infected one. After certain time, the infected individuals recover and return to the susceptible state. In the SIR model, an individual has three possible statuses: susceptible, infected, and refractory (i.e. removal). The infected individuals cannot go back to the susceptible status but can become refractory, which describes the phenomenon of long-time immunity. In addition to these two typical models, there are many other models, such as the SI (susceptible-infected) model and the SIRS (susceptible-infected-refractory-susceptible) model, and the sub-optimal immunity model which lies in between the SIS and the SIR models. The sub-optimal immunity, nonlinear incidence and saturated recovery rate is one of the major concerns in this report. Examples of this kind of diseases include Pertussis (temporary immunity) and Influenza (partial immunity). This kind of models is less studied in comparison to the existing SIS and SIR models.

It should be noted and known that in order to develop a more functioning epidemic model, in the recent years, many intensive research have been carried out especially the incident rate (Xue and Wang, 2012; Zhao and Jiang, 2013; Fan *et al.*, 2012). Some of the reports obtained show that the model exhibits a backward bifurcation (Xue and Wang, 2012; Yousef and Salman, 2016).

Besides this, predator - prey is also another area in mathematical modelling. Ecosystem in biology made us understand that predator is a living organism that devours another living organism, while the prey is the organism which the predator devours. For instance, predator and prey are fox and rabbit, respectively. Predator may devour or not devour their prey prior to feeding on it. However, the reaction of predation usually results in the death of the prey. Predator always has the effect of inhibiting the prey in highly enriched environments, ultimately leading to extinction of the prey population.

In view of that, mathematical study of predator-prey model has been an active research among researchers in various fields. In this area, different researchers from different field such as applied mathematical modelling and ecology are working in this area (Kant and Kumar, 2016).

In recent years, the research in these ODEs epidemic and predator-prey modelling had been shifted to fractional differential equations model. Various fractional epidemic and predator-prey models have been studied (Atangana and Botha, 2013; Ahmed and Elgazzar, 2007; Al-Khaled, 2015; Ameen and Novati, 2017; Angstmann *et al.*, 2017a). In a wider context, the compartmental models which including epidemic models, pharmacokinetics, and in-host virus dynamics were discussed in (Angstmann *et al.*, 2017a).

On the other hand, the well-known Caputo fractional derivative (defined by Michele Caputo in 1967) and the famous Riemann-Liouville fractional integral are the main subjects of many studies in fractional calculus (Podlubny, 1998; Baleanu *et al.*, 2014; Herrmann, 2011). The research work in this area is under a huge development which includes the study of theory of fractional calculus (Diethelm and Ford, 2002; Baleanu *et al.*, 2013), efficient numerical schemes (Bhrawy *et al.*, 2015; Al-Khaled, 2015; Rahimkhani *et al.*, 2017) and application on physical problem (Caputo and Fabrizio, 2015). In addition to that, the fractional derivative is used to increase the stability region of the system, which is more suitable than integer order in an epidemic model with vaccination strategies that lead to multiple points (Rostamy and Mottaghi, 2016; El-Saka, 2013).

1.2 Problem statement

Unfortunately, some evidences show that there are some draw backs in the integer order differential equations (Caputo, 2014; Arenas *et al.*, 2016). In this project, the primary objective of this research and among others is to extend the SIR epidemic and predator- prey models into fractional order models.

Therefore, there is a need to do a research on fractional order model because it increases stability region of system. It is an advantageous approach, which has been used to study the behaviour of disease (like pertussis and influenza) because the fractional order is a generalization of integer order differential equations. The integer order is local in nature while the fractional order is global in nature (El-Saka, 2013; Rostamy and Mottaghi, 2016). However, it is known that fractional order model is locally asymptotically stable, while the integer order model is shown to be stable equilibrium. Fractional order may be considered as a convenient model to describe epidemic problem because of its behaviour.

Particularly, the concept of fractional derivative is more acceptable for modelling certain real world problem than regular derivative (Al-Salti *et al.*, 2016). Fractional derivative helps us in reducing the errors surfacing from the neglected parameters in modelling real life phenomena (El-Saka, 2013). Apart of that, some works on the numerical scheme which claimed that the schemes are able to solve the fractional order modelling high accuracy, such as implicit Adams method (Ameen and Novati, 2017), stochastic process (Angstmann *et al.*, 2016), homotopy analysis method (HAM) (Arqub and El-Ajou, 2013), and Laplace - adomian decomposition method (Rida *et al.*, 2016). Fractional order model provides better agreement with the real data than the integer models because the application of fractional order models with many problems is justified (Arenas *et al.*, 2016).

1.3 Significance of study/research gap

Fractional modelling can describe the behaviour of disease. It serves as a tool to predict, understand and develop strategies to control the spread of infectious disease at different time.

Besides, this epidemiology modelling helps in planning, implementing and evaluating various prevention therapy and control programs. This project is to generate a numerical stability, that is, numerical solutions that will correspond to any of the solutions of the original system of differential equations. Most common numerical inabilities involve the introduction of dynamical features, which include period-doubling bifurcations, chaos, divergence of solution, etc., which are not consistent with the dynamics of the original continuous time model being discretized (Herrmann, 2011).

From the epidemiological point of view, some effects of the fractional-order α on the behavior of dynamical systems of the epidemic model is very significant because the interpretation shows a longer periodic which infected persons can effect the health system (Rostamy and Mottaghi, 2016).

The convergence speed of nearby trajectories are increase with the decrease of α, and as a result can stabilize the stable fixed point. This is of course important in the biotic resource enrichment on a predator-prey population. That is, decreasing the parameter of α has the effect of decreasing the growth rate of population of predator with respect to population of prey. However, the fractional-order system is achieved in the steady state when parameters which affect the value of α are controlled well (Rostamy and Mottaghi, 2016).

Lastly, Malaysia and the rest of the world would again benefit from the research to assess epidemic risks and plan relevant management in order to make early prevention and improvement programme.

1.4 Research objectives

The objectives of this project include:

1. to extend the model of predator-prey to a fractional order system and to perform stability analysis,
2. to extend the model of epidemic with sub-optimal immunity, nonlinear incidence and saturated recovery rate to a fractional order system and to perform stability analysis,
3. to investigate Hopf bifurcations for the both fractional order systems by using successor function method.

1.5 Scope of study

In this project, the model of predator-prey and model of epidemic with sub-optimal immunity, nonlinear incidence and saturated recovery rate are extended to fractional order models by using a system of fractional ordinary differential equations in the sense of the Caputo derivative of order $\alpha \in (0, 1]$. Detailed analysis of the equilibrium points of the models is derived by applying the fractional derivative. Maple 18 is used to obtain the eigenvalues for the corresponding equilibrium points and to perform the stability analysis, where the equilibrium points shall be classified either stable or unstable according to their appearance in the stability region of the fractional order system. In another part, in the case of predator - prey model, a predator-prey model in an environment enriched by a biotic resource to discover how predator always has the effect of inhibiting the prey in highly enriched environments, ultimately leading to extinction of the prey population is studied. Hopf-type bifurcation to further illustrate the behaviour of the system is also presented. In the case of epidemic model, analytically, a certain threshold value of the basic reproduction rate R_0 is obtained. That is, the number of people that one sick person will infect (on average). R_0 is used to establish the existence and stability conditions of both disease-free and endemic for the equilibrium point. On how the value of $\alpha \in (0, 1]$ can increase or control the stability region of the equilibrium points when choosing appropriate values of the fractional order $\alpha \in (0, 1]$ is also investigated. Finally, Adams-type predictor-corrector method is applied, which is an implicit formula method to confirm the analytical results by some numerical simulations for both predator - prey and SIR epidemic models. The results are obtained by choosing appropriate values for $\alpha \in (0, 1]$ and Maple 18 is used as computational platform.

CHAPTER 2

LITERATURE REVIEW

2.1 Research background

Several definitions of fractional-order derivatives have been given by several authors and the commonly used is the Riemann - Liouville definition and the Caputo definitions. It has been proved that Caputo derivative only requires initial conditions by means of integer-order derivative, representing well-understood features of physical situation, which, however, making it more applicable to real world problems (Huang *et al.*, 2017). In this chapter, the Caputo definition is addressed and some related works which related to this project were presented. This project is based on the Caputo derivative.

Definition 2.1

The fractional integral with fractional order $\beta \in \Re^t$ of function $x(t), t > 0$ is defined as:

$$I^{\beta} x(t) = \int_0^t \frac{(t-s)^{\beta-1} x(s)}{\Gamma(\beta)} \mathrm{d}s, \tag{2.1}$$

where $t = t_0$ is the initial time and $\Gamma(.)$ is the Euler's gamma function.

Definition 2.2

The Caputo fractional derivative with order $\alpha \in n-1, n$ of function $x(t), t > 0$ is defined as:

$$cD_t{}^{\alpha} x(t) = I^{n-\alpha} D^n x(t), D_t = \frac{d}{dt}. \tag{2.2}$$

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