A NONSTANDARD OPTIMAL CONTROL PROBLEM ARISING IN AN ECONOMICS APPLICATION: FOR ROYALTY PAYMENT WITH PIECEWISE FUNCTION

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For my beloved father and mother...

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ABSTRACT

A current Optimal Control (OC) problem in the region of financial aspects has numerical properties that do not fall into the standard OC problem formulation. In this study, the state value at the final time is $y(T)=z$ where it is free and a priori unknown. Furthermore, the Lagrangian integrand in the functional is a piecewise constant system of the unknown value $y(T)$. This is not categorized, as in a standard OC problem, and cannot be settled by utilizing Pontryagin's Minimum Principle with the standard boundary conditions at the final time. In the standard case, a free final state value $y(T)$ yields a necessary boundary condition $p(T)=0$ where $p(t)$ is the costate variable. Since the integrand is an element of $y(T)$, the new necessary condition is that $y(T)$ ought to be equivalent to a certain integral that is a continuous system of $y(T)=z$. This study presents a continuous approximation of the piecewise constant integrand function by utilizing a hyperbolic tangent (tanh) approach, and solves a case utilizing a C++ shooting algorithm with a Newton iteration to take care of the Two-Point Boundary Value Problem (TPBVP). The minimizing free value $y(T)$ is computed in an outer loop iteration utilizing the Golden Section Search algorithm. At the end of the study, a comparative discrete-time nonlinear programming (NLP) results are also presented.

ABSTRAK

Masalah Kawalan Optimum (KO) semasa di aspek kewangan mempunyai ciri-ciri berangka yang tidak mengikuti formulasi masalah KO piawai. Dalam kajian ini, nilai keadaan pada masa akhir adalah $y(T)=z$ yang nilai tersebut adalah nilai bebas dan tidak diketahui secara priori. Tambahan pula, kamiran Lagrangian dalam fungsinya adalah sistem malar cebisan nilai $y(T)$ yang tidak diketahui. Hal ini tidak dikategorikan sebagai masalah KO piawai dan tidak boleh diselesaikan dengan menggunakan Prinsip Minimum Pontryagin dengan syarat sempadan piawai pada masa akhir. Dalam kes piawai, nilai keadaan akhir bebas $y(T)$ menghasilkan syarat sempadan $p(T)=0$ yang $p(t)$ adalah pembolehubah ko-keadaan. Oleh kerana kamiran tersebut adalah elemen $y(T)$, syarat keperluan baru yang diperlukan adalah $y(T)$ sepatutnya bersamaan dengan suatu kamiran tertentu yang merupakan sistem berterusan $y(T)=z$. Kajian ini membentangkan penghampiran berterusan fungsi kamiran malar cebisan dengan menggunakan pendekatan tangen hiperbolik (tanh) dan menyelesaikan kes menggunakan algoritma meluru C++ dengan lelaran Newton untuk mengurus Masalah Nilai Sempadan Dua Titik (MNSDT). Peminimuman nilai bebas $y(T)$ dikirakan dalam lelaran gelung luar yang diaplikasikan dalam algoritma Pencarian Keratan Emas. Pada akhir kajian ini, hasil pengaturcaraan tak linear masa diskret juga dibentangkan.

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LIST OF SYMBOLS AND ABBREVIATIONS

λ	-	A parameter that defines the speed of learning
m_0	-	Asymptote of the learning curve
C^n	-	Class of a function with *n*-derivative
c	-	Constraint
δ	-	Difference
r	-	Discount factor
η	-	Natural boundary condition
H	-	Hamitonian function
μ	-	mu
J	-	Objective function or performance index
∂	-	Partial differential
ϕ	-	phi
ψ	-	psi
q	-	Quadratic function
ρ	-	Royalty
S	-	Scrap function
h	-	Step size
τ	-	tau
T	-	Terminal time
c_0	-	The component of unit cost that is subject to learning curve
$u(t)$	-	The control variable for continuous time *t*
$p(t)$	-	The costate variable for continuous time *t*
α	-	The price elasticity of demand
θ	-	theta

$y(t)$	-	The state variable for continuous time *t*
$u_i(t)$	-	The control variable for discrete time *t*
$p_i(t)$	-	The costate variable for discrete time *t*
$y_i(t)$	-	The state variable for discrete time *t*
t	-	Time
υ	-	upsilon
ζ	-	zeta
AMPL	-	A Mathematical Programming Language
BVP	-	Boundary Value Problem
CoV	-	Calculus of Variations
IVP	-	Initial Value Problem
MINOS	-	Modular In-Core Nonlinear Optimization System
NLP	-	Nonlinear Programming
NUDOCCCS	-	Numerical Discretization Method for Optimal Control Problems with Constraints in Controls and States
OC	-	Optimal Control
ODE	-	Ordinary Differential Equation
PMP	-	Pontryagin's Minimum Principle
R&D	-	Research and Development
TPBVP	-	Two-Point Boundary Value Problem

LIST OF PUBLICATIONS

PAPER	TITLE
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CHAPTER 1

INTRODUCTION

This chapter focuses on introducing the research, where the optimization studies will be explained briefly. The title of the research is “A Nonstandard Optimal Control Problem Arising in an Economics Application: For Royalty Payment with Piecewise Function”, which concerns the nonlinear programming (NLP) problem. Therefore, Section 1.1 starts with a brief introduction with three subsections which explained the Calculus of Variations (CoV), Optimal Control (OC), and Initial and Boundary Value Problem. Then, Section 1.2 through to Section 1.6 briefly explain the background of the research, the problem statement, the objectives of the research, the scope of the research, and the significance of the research, respectively. Next, Section 1.7 ends the first chapter with a brief explanation of the thesis outline.

1.1 Introduction

Nowadays, young researchers interested in OC problem by relating with economics application. The economics in OC problem is solved by using the optimization method and NLP approach. The problems can be applied to a real-world scenario. The classical OC problem involves a dynamic system of $\dot{y}(t) = g(t, y(t), u(t))$ where $t \in [0, T]$ and $y(0) = y_0$. In this system, t represents time, $y(t) \in R_n$ denotes the state of the system at time t, $u(t) \in R_r$ is the control at time t, $y_0 \in R_n$ is a given initial state, $T > 0$ is a given terminal time, and $g : R \times R_n \times R_r \to R_n$ is a given function. The interval $[0, T]$ is called the time horizon for the system.

The OC problem in this study will be totally focused on the non-classical (nonstandard) OC problem, where the integral of the performance index depends on the final value of the state variable, $y(T)$. In addition, the costate value, $p(t)$ is not equal to zero at the final time, *T*. This differs to the classic OC theory, where the costate value, $p(t)$ is equal to zero at the final time, *T*.

This study will show how to solve this problem using the NLP approach, and solve a royalty problem which consists of a piecewise function. The performance index depends on the piecewise function and the problem was proposed by (Zinober & Kaivanto, 2008). Royalty is defined as a payment made by clients or developers to the owner for the use of their property. The royalty payment made to the legal owner of the property such as computer software, copyrights, trademarks and patent.

The problem will be explained carefully in five chapters and the contents of each chapter will be explained briefly in the last section of this chapter.

1.1.1 Calculus of Variations

In the seventeenth century, CoV and OC were said to begin with the "brachistochrone curve problem" proposed by Johann Bernoulli (Erlichson, 1999) in 1696. It attracted the attention of Jakob Bernoulli (Erlichson, 1999) and the Marquis De L'Hôpital (Tomlin, 2005) to proceed with the research, but Leonhard Euler (Erlichson, 1999) was the first to elaborate on this matter. Then, Euler and Lagrange (Ferguson, 2004) worked together on the foundation of mechanics and came out with the Euler-Lagrange Equation.

Later, the study was continued by Isaac Newton and Gottfried Leibniz (Ferguson, 2004). Newton used the variational methods to verify the shape of a body that minimizes drag (Tomlin, 2005). In the nineteenth century, Hamilton, Dirichlet, and Hilbert (Brunt, 2004; Olver, 2012) were interested in the research and continued the studies.

CoV has drawn a lot of attention from many famous mathematicians (Gelfand & Fomin, 1963; Tomlin, 2005) and in modern times it continues to gather interest among young researchers. We are also witnessing major theoretical developments in a wide range of applications, such as in physics, engineering, and all branches of mathematics (Olver, 2012). The CoV and the Hamiltonian function will be discussed and used in this study to solve the nonstandard OC problem.

The minimization and maximization techniques are some of the formulations that are widely used in the field of mathematical modelling; used mostly in the engineering, marketing, and economic sectors, it can also be used in conjunction with the CoV (Olver, 2012; Zarebnia & Aliniya, 2011). The CoV provides the mathematical theory for solving the extremizing functional problem where the given functional has a stationary value, either minimum or maximum (Gelfand & Fomin, 1963; Pinch, 1993). Therefore, CoV determines the functions $y(t)$ which minimize or maximize the functional $J[y(t)]$ and return it as a scalar value. The functional is a "function of a function" where a functional $J[y(t)]$ is assigned to each function $y(t)$ in a certain class C^2 a unique real number (Kirk, 1970).

An early problem regarding CoV was to establish the shortest path joining two points with a straight-line segment (Cruz, Torres & Zinober, 2010). Another problem involving the minimization of an integral was proposed in the seventeenth century as a competition and was solved by Jean Bernoulli (Erlichson, 1999) in 1696. The problem was called "the brachystochrone problem", coming from the Greek: *brachystos* = shortest and *chromos* = time. The problem involves a bead sliding under gravity along a smooth wire joining two fixed points, A and B (see Figure 1.1).

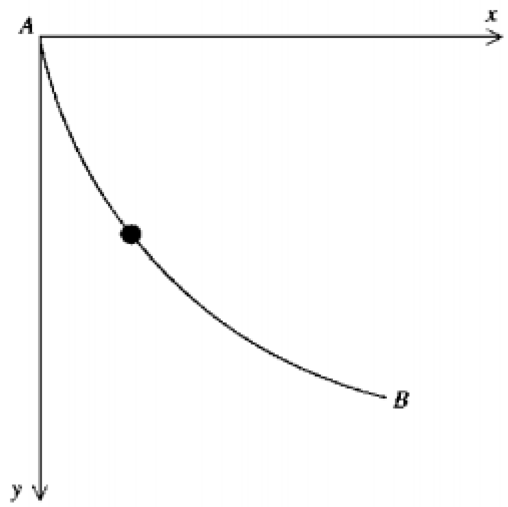

Figure 1.1: Brachistochrone problem: path of least time. Duplicated from Pinch (1993).

In the general case, $y(t) \in R_n$ will be n-differential equations of certain function. The boundary conditions are specified as

$$y(t_0) = y_0 \text{ and } y(T) = y_1 \text{ or unspecified}$$

where $[t_0, T]$ is the time interval and $J : [t_0, T] \times R_n \times R_n \to R$ is a given function, y_0 is the initial value, and y_1 is the final value in R_n.

Theoretically, the difference between CoV and calculus is that CoV deals with functionals, while ordinary calculus deals with functions. The CoV is a theory to minimize a functional by first defining a class of admissible variations and then examining the effect upon the value of the functional J of small variations in the curve along which it is evaluated (Gelfand & Fomin, 1963; Pinch, 1993; Brunt, 2004; Zinober, 2010). The solution of a problem depends on the class of admissible functions (Zinober, 2010; Zinober & Sufahani, 2013).

1.1.2 Optimal Control

OC was well illustrated from the approach made by Dido, the first Queen of Carthage. She asked her servants to take as much land as they could and cover it with ox-hide (Tomlin, 2005). The ox-hide was cut into tiny strips and they tried to enclose the maximum area in a circle of the appropriate radius using a closed curve of given length (Tomlin, 2005). Later, Toneli and Euler proceeded with further research on the problem that was started by Dido and classified it as an "iso-perimetric problem" (Tomlin, 2005).

In 1950, OC began to be studied with interest regarding the optimization problem, and was used in various sectors such as production, finance, economic, marketing, and others. OC consists of choosing from amongst all admissible control variables $u(t)$, with control $u^*(t)$ taking the dynamical system from some initial state $y(t_0)$ at time t_0 to some terminal state $y(T)$ at some terminal time T, in a way that achieves a maximum or minimum of a certain performance index, which is also called the objective functional or cost functional (Kirk, 1970; Berkovitz, 1974; Lewis, 1986; Pinch, 1993; Betts, 2010; Zinober, 2010). The cost performance index is a function of the control and state variables (Zinober, 2010).

1.1.3 Initial and Boundary Value Problem

The problem involved in this study is a differential equation of $\dot{y}(t)=u(t)$ where $y(t)$ is the state variable and $u(t)$ is the control variable. The initial state condition is known as $y(0)=0$ and the final state value $y(T)$ is free and unknown. In this study, the time setting is $[0,10]$ and the value of final state $y(T)$ will be determined by using minimization technique which is Golden Section Search method and NLP approaches (Betts, 2010) which are Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson approximations. These will be run by applying C++ and AMPL program language and further explanation with the example will be discussed in Chapter 4.

1.2 Background of the research

Based on the explanations of CoV in Section 1.1.1 and OC in Section 1.1.2 together with Initial and Boundary Value Problem explanation in Section 1.1.3, the background of this research will now be discussed. As mentioned in Stryk and Bulirsch (1992), Kaya and Noakes (2008), and Betts (2010), there are two approaches to OC computational techniques: the direct and indirect methods. In our study, both techniques are used; however, the indirect method, which involves the Two-Point Boundary Value Problem (TPBVP), and the shooting method will be the main foci. A simple study was done on the direct method in order to compare the results of NLP with the implementation of various discretized accurate schemes (the Euler, Runge-Kutta, Trapezoidal, and Hermite-Simpson methods) along with the optimization algorithm. Then, the study continued with the application of the TPBVP and shooting method for solving the OC problem. Both techniques are used in the validation process.

Furthermore, the recent shooting method that is widely used in the TPBVP is derived from a combination of a numerical technique and the Newton iteration method. This has led us to apply a minimization technique to the shooting method. Instead of using the combination above, a combination of a numerical technique and a minimization technique (the Golden Section Search method) was used. The Golden Section Search method is a one-dimensional minimization technique, which finds the minimum of a function of one independent variable (Press *et al.*, 2007). Both methods, along with the shooting method, will be discussed in more detail in Chapter 3.

This research extended Spence (1981) economic model and involved the royalty function that is in piecewise constant function formulation. Since the problem involved is considered as a nonstandard OC problem, thus, the final state value is unknown and royalty function is depend on this value. The proposed methodologies tested to constant, two-stage and three-stage piecewise constant function that will be converted into continuous approximation of hyperbolic tangent (tanh) function. This is to make it differentiable everywhere when the level of royalty change to a higher stage.

1.3 Problem statement

Recently, an OC problem in the area of economics has been categorized as an issue that does not fall into the standard OC problem formulation. In this economics problem, a firm struggling with intensively low demand for its product will purposely increase the marginal cost by reducing the production of the product and increasing the selling price. Therefore, the requirement to pay a flat-rate royalty on a sale has the effect of increasing the marginal cost, thereby decreasing the output whilst simultaneously increasing the price (Spence, 1981). In this case, the state at the final time, $y(T)=z$ is free and unknown, and additionally the integrand is a piecewise constant function of the unknown value $y(T)$. This is not a standard OC problem and cannot be solved using Pontryagin's Minimum Principle with the standard boundary conditions at the final time.

In the standard problem, a free final state $y(T)$ yields a necessary boundary condition $p(T)=0$, where $p(t)$ is the costate and the integrand does not depend on the final state value $y(T)$. Here, the integrand is a function of the final state value $y(T)$, and the new necessary condition of the final state value $y(T)$ should be equal to a certain integral that is a continuous function of $y(T)=z$. The problem cannot be solved using the classical method because the classical setting does not depend on $y(T)$. The necessary condition in the Hamiltonian function should include the

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