ANALYTICAL SOLUTION FOR MHD FLOW OF UNSTEADY VISCOELASTIC FLUID ARISING IN THE WIRE COATING PROCESS INSIDE A CYLINDRICAL DIE

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The modified homotopy perturbation method (HPM), incorporating He's polynomial and Laplace transform is used to solve incompressible, magnetohydrodynamic (MHD) flow through a cylindrical die for unsteady viscoelastic fluids. The solution is presented as a sum between the steady and unsteady solution for small and large times. For large times, the unsteady solution reduces to the well-known periodic solution that is independent to the initial condition.

Key words: Wire coating; Cylindrical die; Unsteady viscoelastic fluid; Laplace transform; He's polynomial; Homotopy perturbation method.

1. INTRODUCTION

Many engineering fluids, e.g. slurries, pastes, polymer solutions, synovial, paints, microfluidics exhibit numerous strange features, e.g. shear loss/thickening and display of elastic effects which cannot be well described by the Navier-Stokes equations. The conventional stress version is described in viscoelastic second-grade non-Newtonian fluid. Due to its ability in effectively taking non-Newtonian effects, it has been the subject of countless research (Jamil et al. 2009; Sahoo and Labropulu 2012; Massoudi and Phuoc 2008; Erdoğan and Imrak 2010; Razafimandimby and Sango 2012 etc.).

Application of wire coating process can be found in lots of engineering devices for the purpose of generating high and low voltage, protection of humans, and processing signals such as cable and telephone wires and in chemical process in which different types of polymer are used. The coating of the metallic wire depends on the geometry of the die, the viscosity of the fluid, the temperature of the wire and the polymer used in the process of coating. Han and Rao (1978) analyzed the problem of wire coating extrusion both experimentally and theoretically, utilizing a pressure-type die. Siddiqui et al. (2009) studied the wire coating extrusion inside a pressure-type die with the flow of third grade fluid using homotopy perturbation method (HPM). Sajjid et al. (2007) discussed the wire coating with Oldroyd 8-constant fluid using the Homotopy Analyses Method (HAM). Shah et al. (2011) analyzed the isothermal flows of unsteady second grade fluid inside wire coating die with oscillating boundary conditions, using the Optimal Homotopy Asymptotic Method (OHAM). Shah et al. (2013) extended the idea in the straight annular die, and obtained the exact solution by the method of separation of variables.
Homotopy perturbation method developed by He (2005) has been modified by some investigators recently, to obtain more accurate results, rapid convergence, and to reduce the amount of computation. Ghorbani (2009) introduced He's polynomials based on homotopy perturbation method of nonlinear differential equations. Khan and Wu (2011) introduced homotopy perturbation transform method (HPTM) which combined the homotopy perturbation method and Laplace transform method. Khan and Smarda (2012) presented a modified version of homotopy perturbation transform method (MHPTM) which is founded on using Laplace transform for fixing third order boundary layer equation on semi-infinite domain. Nazari-Golshan (2013) produced an altered homotopy perturbation transform method using Fourier transform method for nonlinear and singular Lane-Emden equations.

The goal of the present paper is to derive analytical solutions for the governing model equation for MHD unsteady flow of a viscoelastic fluid of second grade arising in coating wire process inside a cylindrical die. The fluid is electrically conducting in the presence of an external uniform magnetic field. The derived solution is more general, a limiting case is obtained for the corresponding solution of second grade fluid for hydrodynamic.

2. STATEMENT OF THE PROBLEM

We consider the unsteady of electrically conducting, incompressible fluid, viscoelastic fluid of second grade inside a cylindrical die. Figure 1 shows the physical configuration. The $z$-axis is taken along the axis of the flow and $r$ is taken perpendicular to $z$. The fluid is permeated by a uniform transverse magnetic field $\mathbf{B}$, applied parallel to $z$-axis. The metallic wire of radius $R_2$ is pulled within the axial direction with velocity $U_2$ inside a stationary roll die of radius $R_1$. The coordinate system is selected in the centre from the wire, where the axial direction is drawn in the direction where the fluid is moving because of the translation of metallic wire.

The continuity and momentum equations for an incompressible fluid are:

\begin{align}
\nabla \cdot \mathbf{u} &= 0, \\
\rho \frac{Du}{Dt} &= \text{div} \mathbf{T} + \rho \mathbf{b},
\end{align}

where $\rho$ is the fluid density, $\frac{D}{Dt}$ is the material derivative, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{b}$ is the body force per unit mass and $\mathbf{u}$ is the velocity vector.

The stress tensor $\mathbf{T}$ for a viscoelastic of second grade fluid is

\begin{align}
\mathbf{T} = -\rho \mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2,
\end{align}
where I is the identity tensor, $p$ is the pressure, $\mu$ is the dynamic viscosity, $\alpha_1, \alpha_2$ are the viscoelastic parameter of second grade and $A_1$ and $A_2$ are the Revlin-Ericksen tensors which are defined by

$$A_1 = L + L^T,$$

$$A_2 = \frac{d}{dt} A_1 + A_1 L + L^T A_1,$$  

where $L = \nabla V$. We assume that the flow is one-dimensional, so that the velocity field is

$$u = [0, 0, w(r,t)].$$

Using Eqs. (1)-(6) and assuming there is no pressure gradient along axial direction, we get the dimensional governing equation of the form

$$\rho \frac{\partial w}{\partial t} = \mu \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + \alpha_1 \left( \frac{\partial^2 w}{\partial t^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) - \sigma B_0^2 w,$$  

where $\sigma$ is the electrical conductivity of the fluid. The corresponding initial and boundary conditions are

$$w(r,0) = 0, \text{ for } 1 \leq r \leq \xi,$$

$$w(R_2,t) = 0, \text{ for all } t \geq 0,$$

$$w(R_1,t) = U_2, \text{ for all } t \geq 0.$$

Introducing the following dimensionless variables

$$r^* = \frac{r}{R_1}, \quad w^* = \frac{w}{U_2}, \quad \alpha^* = \frac{\alpha_1}{\mu R_1^2}, \quad \tau^* = \frac{\mu}{\rho R_1^2}, \quad M^* = \frac{\sigma B_0^2}{\mu R_1^2},$$

in Eq. (7) and dropping $\ast$, we obtain

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \alpha \left( \frac{\partial^2 w}{\partial t^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) - \frac{\partial w}{\partial t} - M^2 w = 0,$$
with initial and boundary conditions
\[ w(r,0) = 0, \text{ for } 1 \leq r \leq \xi, \]
\[ w(\xi,t) = 0, \text{ for all } t \geq 0, \]
\[ w(1,t) = 1, \text{ for all } t \geq 0, \]
where \( \xi = \frac{A_1}{A_\xi} > 1 \)

3. THE MODIFIED HPM

Consider the following differential equation:
\[ F(w(r,t)) = g(r), \quad r \geq 0, \quad t \geq 0. \]  
(12)

Decomposed \( F \) into two parts, a linear operator \( R \) and a non-linear \( N \)
\[ R(w(r,t)) + N(w(r,t)) = g(r). \]  
(13)

Basing on the homotopy perturbation idea, we construct a homotopy as follow:
\[ R(w(r,t)) + pN(w(r,t)) = g(r), \]  
(14)

considering the linear operator \( R \) in Eq. (13), generate a series expansion with respect to the embedding parameter \( p \)
\[ R(w(r,t)) = R \left( \sum_{i=0}^{\infty} p^i u_i \right), \]  
(15)

and for the nonlinear operator \( N \) in Eq. (13), generate the He's polynomial, \( H_n \), as follows:
\[ N(w(r,t)) = \sum_{n=0}^{\infty} p^n H_n, \]  
(16)

where, the He's polynomial (Ghorbani 2009), \( H_n \), are defined as:
\[ H_n = \frac{1}{n!} \frac{d^n}{dp^n} \left[ \sum_{i=0}^{n} p^i w_i \right] \bigg|_{p=0}. \]  
(17)

Substituting Eqs. (15) and (16) into Eq. (13) we obtain:
\[ R \left( \sum_{i=0}^{\infty} p^i u_i \right) + \sum_{i=0}^{\infty} p^{i+1} H_i = g(r). \]  
(18)

Taking the Laplace transform of both sides of equation (18) we obtain:
\[ L \left[ R \left( \sum_{i=0}^{\infty} p^i u_i \right) \right] + L \left( \sum_{i=0}^{\infty} p^{i+1} H_i \right) = L(g(r)). \]  
(19)

Eq. (19) can be rewritten in the form:
\[ \sum_{i=0}^{\infty} p^i L \{ R(u_i) \} + \sum_{i=0}^{\infty} p^{i+1} L \{ H_i \} = L\{g\}. \]  
(20)

Using Eq. (20) we introduce the recursive relation:
\[ L\{R(u_i)\} = L\{g\}, \]  
(21)

implies
\[ \sum_{i=1}^{\infty} p^i L \{ R(u_i) \} + \sum_{i=0}^{\infty} p^{i+1} L \{ H_i \} = 0. \]  
(22)

The recursive equation deduced from Eq. (22) can be rewritten as:
4. APPLICATION OF MODIFIED HPM

Laplace transform of Eq. (10) is
\[ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} = -s \left[ \alpha \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) - \bar{w} \right] + M^2 \bar{w}, \] (24)

where \( \bar{w}(r,s) = \int_0^s w(r,t) \exp(-st) \, dt. \)

Substituting the recursive Eq. (22) into Eq. (24) leads to the following equation:
\[ \sum_{n=0}^\infty p^n \left( \frac{\partial^2 \bar{w}_n}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}_n}{\partial r} \right) = \sum_{n=0}^\infty p^{n+1} \left(-s \left[ \alpha \left( \frac{\partial^2 \bar{w}_n}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}_n}{\partial r} \right) - \bar{w} \right] + M^2 \bar{w} \right) \] (25)

The recursive equation deduced from Eq. (25) can be written as follows:
\[ p^0: \frac{\partial^2 \bar{w}_0}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}_0}{\partial r} = 0, \]
\[ w_0(\xi, \eta) = 0, w_0(1, \eta) = \frac{1}{\eta}, \]
\[ p^1: \frac{\partial^2 \bar{w}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}_1}{\partial r} = -s \left[ \alpha \left( \frac{\partial^2 \bar{w}_0}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}_0}{\partial r} \right) - \bar{w}_0 \right] + M^2 \bar{w}_0, \] (26)
\[ w_1(\xi, \eta) = 0, w_1(1, \eta) = 0, \]
\[ p^2: \frac{\partial^2 \bar{w}_2}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}_2}{\partial r} = -s \left[ \alpha \left( \frac{\partial^2 \bar{w}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}_1}{\partial r} \right) - \bar{w}_1 \right] + M^2 \bar{w}_1, \]
\[ w_2(\xi, \eta) = 0, w_2(1, \eta) = 0. \]

The transform solution of the recursive equation, Eq. (26), can be expressed by:
Using Maple symbolic code, the inverse Laplace transform of the recursive equation Eq. (27) are:

\[
\begin{align*}
\tilde{w}_0(r,s) &= \frac{1}{s} \left( 1 - \frac{\log(r)}{\log(\xi)} \right), \\
\tilde{w}_1(r,s) &= \left( M^2 + s \right) \left\{ \frac{(-1 + r^2) \log(\xi) (1 + \log(\xi)) + \log(r) (1 - \xi^2 + \log(\xi) - r^2 \log(\xi))}{4s \log(\xi)^2} \right\}, \\
\tilde{w}_2(r,s) &= \frac{1}{128 \log(\xi)} \left\{ \frac{(-1 + r^2) \log(\xi) (1 + \log(\xi)) + \log(r) (1 - \xi^2 + \log(\xi) - r^2 \log(\xi))}{4 \log(\xi)^2} \right\} \delta(t), \\
\tilde{w}_2(r,t) &= \frac{1}{64 \log(\xi)} \left\{ M^2 \left( (-1 + r^2) \log(\xi) (1 + \log(\xi)) + \log(r) (1 - \xi^2 + \log(\xi) - r^2 \log(\xi)) \right) \delta(t) \right\}, \\
\tilde{w}_2(r,t) &= \frac{1}{64 \log(\xi)} \left\{ M^2 \left( (-1 + r^2) \log(\xi) (1 + \log(\xi)) + \log(r) (1 - \xi^2 + \log(\xi) - r^2 \log(\xi)) \right) \delta(t) \right\}, \\
\end{align*}
\]

where \( \delta(t) \) denotes Dirac delta function and \( \delta'(t) \) is the derivative of \( \delta(t) \).

Thus, the modified homotopy solution correct up to second order is given by
It should be noted that in the first two line of Eq. (29) is the steady solution which could be obtained for large times. In the absence of magnetic field (i.e. $M = 0$), the steady solution of Eq. (29) agrees with the solution obtained by by Shah et al. [2013, Eq. (21)].

5. ANALYSIS OF RESULT

The expression for the velocity field $w(r,t)$, given by Eq. (29), are shown graphically in Figs. 2-5 for the dimensionless parameters $r$, $t$ and $M$. Fig. 2 show axial velocity field for different values of time $t$ and radius $r$ when $M = 0$ and $a = 0.5$. One can see from this figure that the velocity field decreases as we increase distance from the centre of the metal wire. Fig. 3 shown the variation of velocity field when $M = 1.7$ and $a = 0.5$. Fig. 4 display the velocity field for different values of $M$. It is noted from both Figs. 3 and 4 that the parameter $M$ has a significant impact on the fluid motion. This is because of the application of magnetic field to an electrically conducting fluid gives rise to resistive force which is known as Lorentz force. This force has tendency to decelerate fluid flow in the boundary layer region.
Fig. 2. Axial velocity field for different values of $r, t$ when $M = 0$. 
Fig. 3. Axial velocity field for different values of \( r, t \) when \( M = 1.7 \)
6. CONCLUSION REMARK

In this work, an analytical solution for the velocity field corresponding to the motion of unsteady viscoelastic second grade fluid in a cylindrical roll die has been derived, using modified homotopy perturbation transform method. This approach seems to be useful and can be used to obtain other analytical solutions for other flow situations (Tao et al. (2013); Noghrehabadi (2013)).

The solution presented as a sum of steady and unsteady solutions. Also, the transient axial velocity tends to the steady-state as the time increases. However, axial velocity is translating in the axial direction against the independent variable $r$ for fixed time $t$ indicate that the transient axial velocity tend to the steady-state whereas the steady-state solutions perform the translating in the axial direction. The result also shows the effects of magnetic field exert a great influence on the general flow pattern.

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References


