On C Integral in the n-Dimensional Euclidean Space

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Abstract
In this paper, we generalize C integral on real line to the n-dimensional Euclidean space. Further, we present the Cauchy criterion for C integrability, the relation between C and Henstock integrals. Finally, we prove that monotone and dominated convergence theorems for the C integral are still valid in the n-dimensional Euclidean space.

Keywords: The n-dimensional Euclidean space; C integral; Cauchy criterion; Monotone and dominated convergence theorems.
1. INTRODUCTION

Bongiorno [1], Bongiorno et al. [2] and Di Piazza [3] gave a constructive minimal Riemann type integral which includes Lebesgue integrable functions and derivatives, so-called C integral on real line R. The problems of the theory of integration are not only on real line, but also in multi dimensions, particularly in the n-dimensional Euclidean space R^n. In general, the problems of multi dimensional integration are quite generalization of the same problems on real line. In this case, the real line is a special case of the n-dimensional Euclidean space. However, not all problems on real line can be automatically generalized to the n-dimensional Euclidean space. For example, if a, b ∈ R and a < c < b, then [a, b] is a union of two non-overlapping intervals [a, c] and [c, b]. However, for the n-dimensional Euclidean space (n ≥ 2) it is not true that if [a, b] ⊂ R^n and a < c < b, then [a, b] = [a, c] ∪ [c, b] [4]. In this paper, we generalize the works of [1-3] to the n-dimensional Euclidean space. Four contributions of this work are as follows: Firstly, the definition of C integral and its basic properties in the n-dimensional Euclidean space are presented. Secondly, Cauchy criterion for C integrability is presented. Thirdly, the relation between C and Henstock integrals is presented. Lastly, we prove that monotone and dominated convergence theorems for C integral are still valid in the n-dimensional Euclidean space.

The rest of this paper is organized as follows. Section 2 describes the fundamental concept of interval and partition in the n-dimensional Euclidean space. Section 3 describes C integral and its basic properties in the n-dimensional Euclidean space. Section 4 the relation between C and Henstock integrals. Section 5 describes monotone and dominated convergence theorems for the C integral in the n-dimensional Euclidean space. Finally, the conclusions of our works are described in section 6.

2. PRELIMINARIES

Let R denote the set of all real numbers. For n ∈ Z^+, the n-dimensional Euclidean space, R^n represents all ordered n-tuple of real number. Thus, R^n can be expressed as a Cartesian product

\[ R^n = R \times R \times \ldots \times R = \{ x = x_1, x_2, \ldots, x_n : x_i \in \mathbb{R}, i = 1, 2, \ldots, n \}. \]

Obviously, R^n is a vector space under vectors scalar product and vectors addition operations and R^n is a Hilbert space under inner product operation.

Let [a, b] ⊂ R be a compact interval on the real line. A cell E in R^n is defined as a non-degenerate interval in R^n, i.e.,

\[ E = \prod_{i=1}^{n} [a_i, b_i], \]
where \( a_i, b_i \in \mathbb{R} \). The volume of a cell \( E \subseteq \mathbb{R}^n \) is the real number \(|E|\) defined by

\[
|E| = \prod_{j=1}^{n} (b_j - a_j).
\]

Evidently, the volume of a cell \( E \) is the Lebesgue measure of \( E \).

Let \( x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \), we can define some norms in \( \mathbb{R}^n \), i.e.,

\[
\|x\|_p = \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty.
\]

\[
\|x\|_\infty = \max \{|x_j| \} , \quad \text{for } p = \infty.
\]

These norms are topologically equivalent.

In this paper, we consider \( \mathbb{R}^n \) as a Banach space endowed with the norm \( \| \cdot \|_\infty \).

Let \( E \) be a cell in \( \mathbb{R}^n \) and \( \delta \) be a gauge (or positive real-valued function) defined on \( E \). For \( x \in E \), an open ball with center at \( x \) and radius \( \delta(x) \) is defined by

\[
B(x, \delta(x)) = \{ y : \|x - y\|_\infty < \delta(x) \}.
\]

Given a cell \( E \in \mathbb{R}^n \). The cell \( E \) can be divided into non-overlapping subcells \( E_1, E_2, ..., E_n \). The division \( \wp = \{E_1, E_2, ..., E_n\} \) of \( E \) is then known as a partition of \( E \). This partition \( \wp \) may then equipped with tags \( \{x_1, x_2, ..., x_n\} \), where \( x_i \in E_i \), for \( i = 1, 2, ..., n \) called tag point of subinterval \( E_i \). The partition

\[
\wp = \{(E_1, x_1), (E_2, x_2) ..., (E_n, x_n)\}
\]

is called a tagged partition if \( x_i \in E_i \), for \( i = 1, 2, ..., n \). A tagged partition of \( E \) is called a McShane \( \delta \)-fine partition of \( E \) if the following conditions are satisfied,

\[
x_i \in E \text{ and } E_i \subseteq B(x_i, \delta(x_i)),
\]

A tagged partition of \( E \) is called a Perron \( \delta \)-fine partition of \( E \) if the following conditions are satisfied,

\[
x_i \in E_i \text{ and } E_i \subseteq B(x_i, \delta(x_i)),
\]

**Lemma 2.1.** Every Perron \( \delta \)-fine partition is a McShane \( \delta \)-fine partition on a cell \( E \).

**Proof.** It follows from the notions of Perron \( \delta \)-fine partition and McShane \( \delta \)-fine partition.

In Section 3, we present \( C \) integrals in the \( n \)-dimensional Euclidean space and its basic properties. All functions in this paper are real-valued functions defined on a cell \( E \).
3. **C INTEGRAL IN THE N-DIMENSIONAL EUCLIDEAN SPACE**

In this section, we present the notions of the C integral in the n-dimensional Euclidean space, its fundamental properties, Cauchy criterion for C integrability and the notion of C primitive of a C integrable function.

**Definition 3.1. (C integral).** A function $f$ is said to be C integrable on a cell $E$, if there is a real number $A$ such that for any $\varepsilon > 0$ there is a gauge on $E$ such that for every McShane $\delta$-fine partition $\wp = \{(E_1, x_1), (E_2, x_2), ... (E_n, x_n)\}$ of $E$ satisfying the condition

$$
(\wp) \sum_{i=1}^{n} \text{dist}(x_i, E_i) < \frac{1}{\varepsilon},
$$

we have

$$
\left| (\wp) \sum_{i=1}^{n} f(x_i)|E_i| - A \right| < \varepsilon.
$$

Here, $(\wp) \sum_{i=1}^{n} f(x_i)|E_i|$ is taken to mean the sum over the McShane $\delta$-fine partition of $E$.

Furthermore, the real number $A$ is called the C integral value of $f$ on $E$ and will be written

$$
A = (C) \int_{E} f.
$$

The collection of C integrable functions on a cell $E$ is denoted by $C(E)$.

**Proposition 3.2.** The following properties are hold in $C(E)$.

(a) Uniqueness, i.e., if a function $f$ is C integrable on a cell $E$, then the C integral value of $f$ on $E$ is unique.

(b) $C(E)$ is a linear space, i.e. if $k$ is a constant and $f$ and $g$ are C integrable on a cell $E$, then so are $cf$ and $f + g$. Moreover

$$
(C) \int_{E} kf = (C)k \int_{E} f \text{ and } (C) \int_{E} f + g = (C) \int_{E} f + (C) \int_{E} g.
$$

(c) Dominated property, i.e. if $f, g \in C(E)$ and $f \leq g$ on a cell $E$, then

$$
(C) \int_{E} f \leq (C) \int_{E} g.
$$

(d) Interval additive property, i.e. if $f \in C(E)$, $f \in C(E)$ and $E = E_1 \cup E_2$ then

$$
f \in C(E) \text{ and } (C) \int_{E} f = (C) \int_{E_1} f + (C) \int_{E_2} f.
$$

If we only want to know whether a function $f$ is C integrable on a cell $E$ without using its C integral value, we may use the following Cauchy’s Criterion for C integrability.
4. CAUCHY CRITERION

Theorem 4.1. (Cauchy Criterion). A function \( f \) is \( C \)-integrable on a cell \( E \) if only if for every \( \varepsilon > 0 \), there is a gauge \( \delta \) on \( E \) such that for every two McShane \( \delta \)-fine partitions \( \varrho_1 = \{(E_1, x_1), (E_2, x_2), \ldots, (E_n, x_n)\} \) and \( \varrho_2 = \{(F_1, y_1), (F_2, y_2), \ldots, (F_n, y_n)\} \) on \( E \), we have

\[
\left| \left( \varrho_1 \sum_{i=1}^{n} f(x_i)|E_i \right) - \left( \varrho_2 \sum_{i=1}^{n} f(y_i)|F_i \right) \right| < \varepsilon.
\]

Proof. \((\Rightarrow)\) Since \( f \) is \( C \)-integrable on a cell \( E \subset \mathbb{R}^n \), then there exist a real number \( A \), such that for every \( \varepsilon > 0 \), there exist a gauge \( \delta \) on a cell \( E \) and for McShane \( \delta \)-fine partition \( \varrho_1 = \{(E_1, x_1), (E_2, x_2), \ldots, (E_n, x_n)\} \) on \( E \) with \( (\varrho_1) \sum_{i=1}^{n} \text{dist}(x_i, E_i) < \frac{1}{\varepsilon} \), the following condition is hold,

\[
\left| \left( \varrho_1 \sum_{i=1}^{n} f(x_i)|E_i \right) - A \right| < \varepsilon.
\]

To this, for every two McShane \( \delta \)-fine partitions

\[
\varrho_1 = \{(E_1, x_1), (E_2, x_2), \ldots, (E_n, x_n)\} \quad \text{with} \quad (\varrho_1) \sum_{i=1}^{n} \text{dist}(x_i, E_i) < \frac{1}{\varepsilon}
\]

and

\[
\varrho_2 = \{(F_1, y_1), (F_2, y_2), \ldots, (F_n, y_n)\} \quad \text{with} \quad (\varrho_2) \sum_{i=1}^{n} \text{dist}(y_i, F_i) < \frac{1}{\varepsilon}
\]

on \( E \), we have

\[
\left| \left( \varrho_1 \sum_{i=1}^{n} f(x_i)|E_i \right) - A \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \left( \varrho_2 \sum_{i=1}^{n} f(y_i)|F_i \right) - A \right| < \frac{\varepsilon}{2}, \text{respectively}.
\]

Hence,

\[
\left| \left( \varrho_1 \sum_{i=1}^{n} f(x_i)|E_i \right) - \left( \varrho_2 \sum_{i=1}^{n} f(y_i)|F_i \right) \right| = \left| \left( \varrho_1 \sum_{i=1}^{n} f(x_i)|E_i \right) - A + A - \left( \varrho_2 \sum_{i=1}^{n} f(y_i)|F_i \right) \right|
\]

\[
\leq \left| \left( \varrho_1 \sum_{i=1}^{n} f(x_i)|E_i \right) - A \right| + \left| \left( \varrho_2 \sum_{i=1}^{n} f(y_i)|F_i \right) - A \right|
\]

\[
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

\((\Leftarrow)\) Assumed that for every positive integer \( n \in \mathbb{Z}^* \), there exist a corresponding gauge \( \delta_n^* \) on \( E \), such that for any two McShane \( \delta_n^* \)-fine partitions
\[ \varphi_1^* = \{(E_1, x_1), (E_2, x_2), \ldots, (E_n, x_n)\} \text{ dan } \varphi_2^* = \{(F_1, y_1), (F_2, y_2), \ldots, (F_m, y_m)\} \]
on E, with
\[
\left( \varphi_1^* \sum_{i=1}^n \text{dist}(x_i, E_i) \right) < \frac{1}{\varepsilon} \text{ and } \left( \varphi_2^* \sum_{i=1}^n \text{dist}(y_i, F_i) \right) < \frac{1}{\varepsilon}
\]
the following condition is hold
\[
\left| \left( \varphi_1^* \sum_{i=1}^n f(x) \right)_{|E_i} \right| < \left( \varphi_2^* \sum_{i=1}^n f(y) \right)_{|F_i} \frac{1}{n} \quad (1)
\]
For every positive integer \( n \in \mathbb{Z}^+ \), we set a positive gauge \( \delta_n \) on \( E \), as follows
\[
\delta_1(x) = \delta_1^*(x), \\
\delta_2(x) = \min\{\delta_1(x), \delta_2(x)\}, \\
\delta_3(x) = \min\{\delta_1(x), \delta_2(x), \delta_3(x)\}, \\
\vdots \\
\delta_n(x) = \min\{\delta_1(x), \delta_2(x), \ldots, \delta_{n-1}(x), \delta_n^*(x)\}.
\]
Thus, \( \delta_n(x) \geq \delta_1(x) \geq \ldots \geq \delta_n(x) \) for every \( x \in E \). Therefore, for two positive integers \( n, m \in \mathbb{Z}^+ \), where \( n > m \), every McShane \( \delta_n \)– fine partition
\[
\varphi_n = \{(E_1, x_1), (E_2, x_2), \ldots, (E_n, x_n)\} \text{ with } \left( \varphi_n \right) \sum_{i=1}^n \text{dist}(x_i, E_i) < \frac{1}{\varepsilon}
\]
is a McShane \( \delta_n \)– partition
\[
\varphi_m = \{(E_1, x_1), (E_2, x_2), \ldots, (E_m, x_m)\} \text{ with } \left( \varphi_m \right) \sum_{i=1}^m \text{dist}(x_i, E_i) < \frac{1}{\varepsilon}
\]
Then, for any \( n \in \mathbb{Z}^+ \), we can take a McShane\( \delta_n \)– fine partition
\[
\varphi_\infty = \{(E_1, x_1), (E_2, x_2), \ldots, (E_n, x_n)\} \text{ on } E \text{ with } \left( \varphi_\infty \right) \sum_{i=1}^n \text{dist}(x_i, E_i) < \frac{1}{\varepsilon}
\]
We construct
\[
A_n = \left( \varphi_\infty \right) \sum_{i=1}^n f(x)_{|E_i}.
\]
To this, we obtain a sequence of real numbers \( \{A_n\} \). Let a positive number \( \varepsilon > 0 \), according to Archimedean property, there exist a positive integer \( n_0 > \frac{2}{\varepsilon} \). Such that from \( m, n \in \mathbb{Z}^+ \)
where \( m, n \geq n_0 \), a gauge \( \delta_n \) on \( E \) and (1), we have

\[
|A_n - A_m| = \left| (\varphi_n \sum f(x_i)|E_i| - (\varphi_m \sum f(x_i)|E_i|) \right| < \frac{1}{m} n_0 < \frac{\varepsilon}{2}.
\]  

(2)

Hence, \( \{A_n\} \) is a Cauchy sequence in \( R \). Since \( R \) is a complete Banach space, then \( \{A_n\} \) is a convergent sequence, say to a number \( A \in R \). Thus for every positive number \( \varepsilon > 0 \) as stated above, there exist a positive integer \( n_1 \in Z^+ \), where \( n \geq n_1 \). Such that for every

\[
|A_n - A| < \frac{\varepsilon}{2}
\]

(3)

From (2) and (3), we get

\[
\left| (\varphi \sum f(x_i)|E_i|) - A \right| = \left| (\varphi \sum f(x_i)|E_i|) - A_n + A_n - A \right|
\]

\leq \left| (\varphi \sum f(x_i)|E_i|) - A_n \right| + |A_n - A|

< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.

In previous paper [5], we presented the relation between \( C \) and Lebesgue integrals. In the next section, we present the relation between \( C \) and Henstock Integrals.

5. \( C \) VERSUS HENSTOCK INTEGRALS

In this section, the relation between \( C \) and Henstock integrals in the \( n \)-dimensional Euclidean space is presented. We firstly present the definition of Henstock integral.

**Definition 5.1. (Henstock integral).** A function \( f \) is said to be Henstock integrable on a cell \( E \), if there is a real number \( A^* \) such that for any \( \varepsilon > 0 \) there is a gauge on \( E \) such that for any Perron \( \delta \)-fine partition \( \varphi = \{(E_{i_1}, x_{i_1}), (E_{i_2}, x_{i_2}), ..., (E_{i_n}, x_{i_n})\} \) of \( E \), we have

\[
\left| (\varphi \sum f(x_i)|E_i|) - A^* \right| < \varepsilon
\]

Here, \((\varphi \sum f(x_i)|E_i|)\) is taken to mean the sum over the \( \delta \)-fine Perron partition \( D \) of \( E \). Furthermore, the number \( A^* \) is called the Henstock integral value of \( f \) on \( E \) and will be written

\[
A^* = (R^*) \int_E f.
\]

The collection of Henstock integrable functions on a cell \( E \) is denoted by \( R^*(E) \).
The following theorem states that the collection of all $C$-integrable functions is properly contained in the collection of all Henstock-Kurzweil integrable functions. This theorem is used to prove the monotone convergence theorem for the $C$-integrals in the $n$-dimensional Euclidean space.

**Proposition 5.2.** Every $C$ integrable function is Henstock integrable on the same cell, with the same value.

Proof. From Lemma 2.1, since every $\delta$-fine Perron partition is a $\delta$-fine McShane partition on a cell $E$ (See), then the proof is clear. \(\Box\)

To show that the converse of Proposition 5.2 is not true, we consider to the following example.

**Example 5.3.** Without lost generality, we present an example of a function defined on real line. Let $F : [0, 1] \to \mathbb{R}$ defined as

$$F(x) = \begin{cases} x \sin x^2 , & \text{for } 0 < x \leq 1 \\ 0 , & \text{for } x = 0 \end{cases}$$

We define a function $f : [0, 1] \to \mathbb{R}$ as $f = F'$ for every $x \in [0, 1]$. The plots of $F$ and $f$ are given in Figures 1 and 2, respectively. Obviously, that $f$ is unbounded on $[0, 1]$, particularly near 0. Hence, $f$ is not Riemann integrable on $[0, 1]$.

![Fig. 1, plot of F on [-0.4, 0.4].](image-url)
We will show that $F$ is a primitive of $R^*$, i.e. $(R^*) \int f = F$. From the Cauchy extension of Henstock integral, we have

$$\int_{[0,x]} f(x) = \lim_{a \to 0} (x \sin x^2 - a \sin a^2)$$

$$= x \sin x^2 - 0$$

$$= x \sin x^2$$

$$= F(x). \quad (4)$$

Next, we show that $f$ is not $C$ integrable on $[0, 1]$. Here, we assume that $f$ is $C$ integrable on $[0, 1]$. It means that for every positive number $\varepsilon > 0$, there exist a gauge $\delta > 0$ on $[0, 1]$. Thus, if we take

$$a_h = (\pi + 2h\pi)^\frac{1}{2} \quad \text{and} \quad b_h = \left(\frac{\pi}{2} + 2hn\right)^\frac{1}{2}, \quad (5)$$

it is easy to understand that intervals $(a_h, b_h), h = 1, 2, 3, \ldots$ are disjoint and

$$\sum_{h=1}^{\infty} a_h = \infty \quad \text{as well} \quad \sum_{h=1}^{\infty} b_h = \infty.$$

From (4) and (5), we get

$$F(a_h) = a_h \sin a_h^2$$

$$= (\pi + 2h\pi)^\frac{1}{2} \sin(x + 2h\pi)$$

$$= (p + 2hp)^\frac{1}{2} \cdot 0$$

$$= 0,$$
and

\[ F(b_h) = b_h \sin b_h^{-2} = \left( \frac{\pi}{2} + 2h\pi \right)^{\frac{3}{2}} \sin \left( \frac{\pi}{2} + 2h\pi \right) \]

\[ = \left( \frac{\pi}{2} + 2h\pi \right)^{\frac{3}{2}} \cdot 1 \]

\[ = \left( \frac{\pi}{2} + 2h\pi \right)^{\frac{3}{2}} \cdot 1. \]

For every \( m, n \in \mathbb{Z}^+ \) we have

\[ (a_{m+i}, b_{m+i}) \subset (0, \delta(0)), (i = 1, 2, 3, \ldots, n), \text{ where } \varepsilon < \sum_{i=1}^{n} a_{m+i} < \frac{1}{\varepsilon}, \]

Hence, \( \sum_{i=1}^{n} b_{m+i} > \sum_{i=1}^{n} a_{m+i} > \varepsilon \). Let, we define intervals

\[ E_1 = (a_{m+1}, b_{m+1}), E_2 = (a_{m+2}, b_{m+2}), \ldots, E_n = (a_{m+n}, b_{m+n}), \]

thus, a collection of \( \{E_1, 0\}, \{E_2, 0\}, \ldots, \{E_n, 0\} \) is a partial McShane \( \delta \)-fine partition in \([0, 1]\), where

\[ \sum_{i=1}^{n} \text{dist}(0, E_i) = \sum_{i=1}^{n} a_i < \frac{1}{\varepsilon}. \]

Furthermore,

\[ \sum_{i=1}^{n} \left| f(0)|E_i| - \mathcal{C} \int_{E_i} f \right| = \sum_{i=1}^{n} |F(b_{m+i}) - F(a_{m+i})| = b_{m+i} > \varepsilon, \]

This is a contradiction with the assumption that \( f \) is \( \mathcal{C} \) integrable on \([0, 1]\).

The relation of \( \mathcal{C} \) and Henstock integrals can be depicted in Figure 3 as follow.

\[
\begin{array}{c}
\{ \text{Collection of } \mathcal{C} \text{ integrable functions} \} \\
\subset \\
\neq \\
\{ \text{Collection of } \mathcal{R}^* \text{ integrable functions} \}
\end{array}
\]

**Fig. 3**, the relation between \( \mathcal{C} \) and Henstock integrals.

In the theory of integration, convergence theorems play an important role in deriving a sufficient condition for the limit of integral values of integrable functions in order to have the same value with the limit of sequence of the functions.
6. CONVERGENCE THEOREMS FOR C INTEGRAL

In this section, we present the monotone and dominated convergence theorems for the C integral in the \( n \)-dimensional Euclidean space as generalization of the same case on real line. We prove that such convergence theorems for the C integral are still valid in the \( n \)-dimensional Euclidean space.

**Theorem 6.1. (Monotone convergence theorem).** Let \( \{f_n\} \) be a sequence of real valued C integrable function on a cell \( E \subset \mathbb{R}^n \). If \( \{f_n\} \) are monotone and convergent to a real valued function \( f \) on \( E \) and \( \lim_{n \to \infty} \int_E f_n \) exists and finite, then \( f \) is C integrable on \( E \) and

\[
\left( \text{C} \right) \int_E f = \lim_{n \to \infty} \int_E f_n.
\]

Proof. Let \( f \) be any C integrable function on a cell \( E \), by Theorem 5.2, \( f \) is Henstock integrable on \( E \) with the same value of the integrals. Furthermore, if \( f \) is a non-negative, \( f \) is Lebesgue integrable on \( E \) with the same value of the integrals [6,7].

Let \( \{f_n\} \) be a sequence of real-valued C integrable functions on a cell \( E \subset \mathbb{R}^n \). First, we prove for the case \( \{f_n\} \) is monotone increasing and convergent to a real valued function \( f \) on \( E \).

Since \( \{f_n\} \) is monotone increasing, then we have

\[
f_1 \leq f_2 \leq f_3 \leq \ldots \leq f_n \leq f_{n+1} \leq \ldots
\]

Consequently,

\[
0 \leq f_2 - f_1 \leq f_3 - f_1 \leq \ldots \leq f_n - f_1 \leq \ldots
\]

By monotone convergence theorem for the Lebesgue integral [6,7], we have

\[
(L) \int_E (f_n - f_1) = \lim_{n \to \infty} (L) \int_E (f_n - f_1)
\]

By Theorem 3.1 in [5], for every \( n \in \mathbb{Z}^+ \), we have

\[
(L) \int_E (f_n - f_1) = (C) \int_E (f_n - f_1) = (C) \int_E f_n - (C) \int_E f_1.
\]

By the hypothesis, that \( \{f_n\} \) is convergent to a real valued function \( f \) on \( E \) and \( \lim_{n \to \infty} \int_E f_n \) exists and finite, then from (6), we have the function \( f - f_1 \) is Lebesgue integrable on \( E \). Therefore, if \( f = (f - f_1) + f_1 \) is Lebesgue integrable on \( E \), then \( f \) is C integrable on \( E \) with the same value of the integrals. Hence
Second, we prove for the case \( \{ f_n \} \) is monotone decreasing and convergent to a real valued function \( f \) on \( E \). Since \( \{ f_n \} \) is monotone decreasing, then
\[
f_1 \geq f_2 \geq f_3 \geq \ldots \geq f_n \geq \ldots
\]
Consequently,
\[
0 \leq f_1 - f_2 \leq f_1 - f_3 \leq \ldots \leq f_1 - f_n \leq \ldots
\]
By monotone convergence theorem for the Lebesgue integral \([6,7]\), we have
\[
(L) \int_E (f_1 - f_n) = \lim_{n \to \infty} (L) \int_E (f_1 - f_n) \quad (7)
\]
By Theorem 3.1 in [5], for every \( n \in \mathbb{N} \), we have
\[
(L) \int_E (f_1 - f_n) = (C) \int_E (f_1 - f_n)
= (C) \int_E f_1 - (C) \int_E f_n
\]
By the hypothesis, that \( \{ f_n \} \) is convergent to a real valued function \( f \) on \( E \) and \( \lim_{n \to \infty} (C) \int_E f_n \) exists and finite, then from (7), we have the function \( f_1 - f \) is Lebesgue integrable on \( E \). Therefore, if \( f = f_1 - (f_1 - f) \) is Lebesgue integrable on \( E \), then \( f \) is \( C \)-integrable on \( E \) with the same value of the integrals. Hence
\[
(C) \int_E f = (C) \int_E f_1 - (C) \int_E (f_1 - f)
= (C) \int_E f_1 - (C) \int_E f_1 + (C) \int_E f
= (C) \lim_{n \to \infty} \int_E f_n, \quad \square
\]

**Theorema 6.2 (Dominated convergence theorem).** Let a cell \( E \subset \mathbb{R}^n \), a sequence of measurable function \( \{ f_n \} \) are \( C \) integrable on \( E \). If

(a) \( \{ f_n \} \) converges to \( f \) almost every where on \( E \)
(b) for every $n \in \mathbb{N}$, $g \leq f_n \leq h$ almost every where on $E$ then $f \mathcal{C}$ integrable on $E$, furthermore

$$(C) \int_E f = \lim_{n \to \infty} \int_E f_n.$$ 

Proof. For every $n \in \mathbb{Z}^+$, from the second hypothesis, if $g \leq f_n \leq h$ almost every where on $E$, then $0 \leq f_n - g \leq h - g$

Since $g$ and $h$ are $C$ integrable on $E$, according to Proposition 3.2 part (b), $h - g$ is $C$ integrable on $E$. According to Proposition 5.2, $h - g$ is Henstock integrable on $E$ with the same value. Furthermore, from [5] if $h - g \geq 0$, then $h - g$ is Lebesgue integrable on $E$ with the same value. From the first hypothesis, the collection of all $C$ integrable functions is properly contained in the collection of all Henstock integrable functions on $E \subset \mathbb{R}^n$. We have proven that the monotone and dominated convergence theorems for the $C$ integral are still valid in the $n$-dimensional Euclidean space. In addition, we propose a number of future research activities related to the $C$ integral. First, we can generalize the $C$ integral in the different domain, such as Banach and Sequence spaces. Second, the properties of Small Riemann Sum (SRS) is still an attract issue for the $C$ integral.

7. CONCLUSIONS

In this paper, we have used a notion of a cell $E \subset \mathbb{R}^n$ in the $n$-dimensional Euclidean space. The $C$ integral in the $n$-dimensional Euclidean space is successfully generalized from real line. Four basic properties of $C$ integral in the $n$-dimensional Euclidean space, i.e., uniqueness, $C(E)$ is a linear space, monotone property and interval additive property are presented. The relation of $C$ and Henstock integrals are presented. The result show that the collection of all $C$ integrable functions is properly contained in the collection of all Henstock integrable functions on $E \subset \mathbb{R}^n$. We have proven that that monotone and dominated convergence theorems for the $C$ integral are still valid in the $n$-dimensional Euclidean space. In addition, we propose a number of future research activities related to the $C$ integral. First, we can generalize the $C$ integral in the different domain, such as Banach and Sequence spaces. Second, the properties of Small Riemann Sum (SRS) is still an attract issue for the $C$ integral.
REFERENCES


