NUMERICAL SOLUTION FOR FRACTIONAL-ORDER LOGISTIC EQUATION

LIYANA NADHIRA BINTI KAHARUDDIN

A thesis submitted in
fulfillment of the requirement for the award of the
Degree of Master of Science

Faculty of Applied Sciences and Technology
Universiti Tun Hussein Onn Malaysia

APRIL 2021

DEDICATION

Every challenging work needs self-effort as well as guidance of elders,
especially those who are very close to our hearts.
I dedicate my humble effort to my sweet and loving parents,
Kaharuddin Yaacob and Sakdiah Mohd Kassim;
my dearest siblings, Muhamad Amzar Hail and Lina Nayli;
last but not least, to Izuan Aiman Ismail.
May Allah bless all of you and shower your life with joy and happiness.

ACKNOWLEDGEMENT

Bismillahirahmanirrahim. First of all, my thanks to Allah s.w.t that, because of His blessing and mercy, I finally managed to complete this research. My appreciation goes to my supervisor, Dr. Siti Suhana binti Jamaian, for her encouragement and guidance throughout the accomplishment of this research.

Secondly, my sincerely gratitude goes to Assoc. Prof. Dr. Phang Chang for guiding me until I finished this thesis. A special thanks goes to a person who have always motivated and inspired me so much in completing this thesis.

Next, I would like to express my deepest gratitude to my beloved parents and siblings for always believing in me. Without their support, I would probably not be able to reach this stage. Last but not least, I would like to thank all the individuals who were involved in the competion of this thesis. May this thesis benefit and be useful for all readers. InshaaAllah.

ABSTRACT

Recently, in the direction of developing realistic mathematical models, there are a number of works that extended the ordinary differential equation to the fractional-order equation. Fractional-order models are thought to provide better agreement with the real data compared with the integer-order models. The fractional logistic equation is one of the equations that has been getting the attention of researchers due to its nature in predicting population growth and studying growth trends, which assists in decision making and future planning. This research aims to propose the numerical solution for the fractional logistic equation. Two different solving methods, which are the Adam's-type predictor-corrector method and the *Q*-modified Eulerian numbers, were successfully applied to two versions of the fractional-order logistic equation, which are the fractional modified logistic equation and the fractional logistic equation, respectively. The fractional modified logistic equation, which involved the extended Monod model, was solved by the Adam's-type predictor-corrector method and was applied in estimating microalgae growth. The results show that the fractional modified logistic equation agreed with the real data of microalgae growth. Meanwhile, a closed-form solution by the *Q*-modified Eulerian numbers was proposed for the fractional logistic equation. These modified Eulerian numbers were obtained by modifying the Eulerian polynomials in two variables. Interestingly, these modified polynomials corresponded to the polylogarithm $Li_p(z)$ of the negative order and with a negative real argument, z. The proposed method via the modified Eulerian numbers can provide the generalised solution for an arbitrary value. The proposed method was shown to achieve numerical convergence. The numerical experiment shows that this method is highly efficient and accurate since the absolute error obtained from the subtraction of the exact and proposed solution is considerably small.

ABSTRAK

Baru-baru ini, ke arah pengembangan model matematik yang realistik, terdapat sejumlah penyelidikan yang memperluas persamaan pembezaan biasa ke persamaan pembezaan pecahan. Model pembezaan pecahan dianggap memberikan kesepakatan yang lebih baik dengan data sebenar dibandingkan dengan model pembezaan integer. Persamaan pembezaan logistik pecahan adalah salah satu persamaan yang mendapat perhatian penyelidik kerana sifatnya dalam meramalkan pertumbuhan populasi dan mengkaji corak pertumbuhan yang membantu dalam membuat keputusan dan perancangan masa depan. Penyelidikan ini bertujuan untuk mencadangkan penyelesaian untuk persamaan logistik pecahan. Dua kaedah penyelesaian yang berbeza berjaya diaplikasikan pada dua versi persamaan logistik pecahan iaitu kaedah peramal-pembetulan jenis Adam dan nombor Eulerian. Persamaan logistik pecahan yang diubah yang melibatkan model Monod lanjutan telah diaplikasikan dalam meramalkan pertumbuhan mikroalga. Hasilnya menunjukkan persamaan logistik pecahan sepadan dengan data eksperimen pertumbuhan mikroalga. Sementara itu, penyelesaian berbentuk tertutup dengan nombor Eulerian yang diubah suai Q dicadangkan untuk versi lain dari persamaan logistik pecahan. Nombor Eulerian yang diubah ini diperoleh dengan mengubah polinomial Euler kepada dua pemboleh ubah. Menariknya, polinomial yang diubah ini sesuai dengan polilogaritma $Li_p(z)$ susunan negatif dan dengan argumen nyata negatif, z. Kaedah yang dicadangkan melalui nombor Eulerian yang diubah dapat memberikan penyelesaian umum untuk nilai arbitrari. Kaedah yang dicadangkan ini didapati mencapai penumpuan berangka. Eksperimen berangka menunjukkan bahawa kaedah ini sangat efisien dan tepat kerana ralat mutlak yang diperoleh daripada hasil tolak penyelesaian tepat dengan penyelesaian yang dicadangkan adalah sangat kecil.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS AND ABBREVIATIONS

$A_k(x)$	-	Eulerian polynomials
β	-	Beta
C_i	-	Saturation constant
D^β	-	Differential operator of the beta order
$E^\beta_{k,Q}$	-	Q -modified Eulerian number
e	-	Exponential function
f	-	Function
!	-	Factorial
Γ	-	Gamma
∞	-	Infinity
$J(Q)$	-	Eulerian polynomial
K	-	Carrying capacity
k	-	Constant
Li_p	-	Polylogarithm
λ	-	Lambda
mg / L	-	Microalgae biomass
μ	-	Specific growth rate
$\mu_{\max}$	-	Maximum specific growth rate
n	-	Constant
R_i	-	Substrate concentration
R_N	-	Concentrations of nitrogen
R_P	-	Concentrations of phosphorus
t	-	Time
x	-	Monomial

$X(t)$	-	Fractional logistic equation
y_m	-	Maximum cell concentration
y_0	-	Initial concentrations of microalgae
$y(t)$	-	Modified logistic equation
FMLE	-	Fractional modified logistic equation
NLSF	-	Nonlinear least-squares fitting
OMLE	-	Ordinary modified logistic equation

LIST OF APPEDICES

CHAPTER 1

INTRODUCTION

1.1 Research background

Ecology is one of the branches in biology that concerns the relationship and the interactions between organisms and their environment. It is often studied together with other sciences, including mathematics, engineering and also social sciences. On that account, it can be said that mathematics has become an important tool in describing the physical system of ecology. Population ecology is the study of population size, density, distribution and changes over time. By performing the study, ecologists are able to gather the data, which can help them predict growth trends and manage the population size of a particular living things. This is essential in improving biodiversity conservation. Population size may increase or decrease, changes may occur quickly or slowly, and the effects on other populations may be marked or slight. Population dynamics refer to the changes in the number of organisms over time, and it is fundamental in ecology. Countless efforts have been made to develop a realistic mathematical model that can describe population dynamics. Among such models is the logistic equation, which was originally introduced by Pierre Francois Verhulst in 1838 (Cushing, 1998).

A population is a group of individuals of the same species that inhabit an area at the same time. The diagnostic features of living matter are growth and reproduction, and a population must be defined on the same basis. Just as living matter must exhibit growth and commonly undergo reproduction, organisms grow and usually bring an assemblage of descendants that together form a population (Pielou, 1974).

Indeed, there are many contexts in which it is important to understand population dynamics. For instance, in fisheries management, the manager is interested in being able to predict the density of fish population under different management plans. An agronomist may wish to know the yield of a population of maize plants when planted at a particular density, while an epidemiologist will want to know the density of disease-infected humans next month (Vandermeer & Goldberg, 2013). These examples clearly demonstrate the significance of modelling population growth. In view of that, the mathematical study of the population growth model has become an active research study among researchers in many fields, especially in the areas of mathematical modelling and ecology. Even though the original application of the logistic equation was in expressing population growth, the application of the equation, especially the fractional-order type, has recently broadened to many other fields, such as medicine, agriculture, economics, business and also physics. The research in the ordinary differential equation for population growth has also recently shifted to the fractional differential equation model.

In the past few decades, the interest in applying the fractional-order equation to describe real-life phenomena has greatly developed. It has been shown that some fractional-order equations describe some complex physical phenomena in a better way (Atangana & Secer, 2013). One of the most important advantages of fractional-order models in comparison with the integer-order ones is that fractional integrals and derivatives are a powerful tool for the description of memory and hereditary properties of some materials (Area, Losada, & Nieto, 2016). In addition, many authors pointed out that fractional-order derivatives and integrals are very appropriate for application in various fields. The research work on this ground is undergoing a huge development in terms of the theoretical study of fractional calculus (Diethelm & Ford, 2002; Zhou, Wang, & Zhang, 2016), efficient numerical methods (Diethelm, Ford, & Freed, 2002; El-Sayed, El-Mesiry, & El-Saka, 2007) and also the application to physical phenomena (Caputo & Fabrizio, 2015; Das & Gupta, 2011). There were researchers who not only proposed the fractional derivative of various models but also provided the validation of the integer- and fractional-order models in comparison with the real data (Abobakr et al., 2017; Dzieli, Sarwas, & Sierociuk, 2011; Freeborn, 2013).

The logistic equation can be considered as an important differential equation and has received attention from researchers around the world due to its ability to describe several biological and social phenomena. In predicting the population growth of various types of living organisms, there are various parameters that may need to be considered. Hence, some researchers modified the logistic equation into population growth models that corresponded to specific living organisms, such as the predator-prey model, initially proposed by Lotka (Lotka, 1925), and the logistic Allee effect model (Allee & Bowen, 1932). Another example of logistic equation modification was by Mohd Sadiq et al. (2018), where the study implemented the extended Monod model in the logistic equation for the prediction of microalgae growth.

With the development in fractional calculus, the extension of the ordinary logistic differential equation to the fractional logistic differential equation model is also one of the attempts in developing appropriate population growth models (Area, Losada, & Nieto, 2016; El-Sayed, El-Mesiry, & El-Saka et al., 2007; Ortigueira & Bengochea, 2017; D'Ovidio & Loreti, 2018; West, 2015). A considerable amount of research works regarding the fractional-order logistic equation has been done to determine exact, analytical and approximation solutions. Among the research works are the variational iteration method by He (1999), the homotopy perturbation method by Sweilam et al. (2007), and the collocation method by Mohamed and Sherif (2013). There are also several research that successfully applied the fractional logistic equation to describe real-world problems, such as the study by Bas and Ozarslan (2018).

Apart from all the successful research works in solving the fractional logistic equation, as far as it is known, the closed-form solution for the fractional logistic equation remains unsolved except for the order $\beta = 1$. Also, research works have not been widely applying the fractional logistic equation to describe real phenomena. Hence, the present research aims to conduct a study that considers both aspects, which are the closed-form solution and the application of the fractional logistic equation.

1.2 Problem statement

In predicting the population growth of various types of the living organisms, there are different parameters may need to be considered. Hence, there were some researchers who modified the logistic equation to become a population growth models that correspond to the specific living organisms such as the model for microalgae growth by (Ummal Aisha Farhana et al., 2018) where the researchers implement the extended Monod model in the logistic equation for the prediction of the microalgae growth.

They modified the ordinary logistic differential equation by involving the extended Monod model and used it to predict the growth of *Botryococcus sp.* microalgae. However, the estimated results from the model seems to not adequately match the experimental data. It is undeniable that the integer-order version of the logistic equation has contributed much in modelling population growth. However, complex relationships, such as the interactions within the food webs or several behaviours that occur in most cases of nature dependencies, may give some errors in the population growth prediction by the integer-order logistic equation. These may be described better by the fractional-order version of the logistic equation, since some evidences show that fractional-order equation produces better real-data conclusions than the models that are structured to depend on the integer-order derivative (Du, Wang, & Hu, 2013; Sweilam, Nagy, & Elfahri 2019). The generalisation of the differential equation to the fractional-order derivative may help to reduce inaccuracies emerging from the ignored parameters in the modelling of real-life problems (Ul et al., 2018). The solution to the logistic equation where the derivative is of arbitrary order has received much attention from researchers in the area of fractional calculus. Several attempts were made to determine the exact, analytical or approximation solution. One of the most well-known numerical method in solving the fractional-order equation is the Adam's-type predictor corrector method. In this study the Adam's-type predictor corrector method is applied to solve the fractional modified logistic equation for microalgae growth.

In solving the fractional differential equation, a closed-form solution is considered important, since there is a high computational cost associated in performing numerical differentiations, which necessitates the derivation of closed-form expressions for algorithm runtime. To the best of the author's knowledge, the closed-form solution for the fractional logistic equation remains unsolved. Hence, in this study, the integer-order logistic equation was extended to the fractional-order logistic equation in two versions, which were the fractional modified logistic equation and the fractional logistic equation. The numerical solution for the fractional modified logistic equation was obtained by applying a proper numerical method, while the closed-form solution for the fractional logistic equation needed to be proposed.

1.3 Research objectives

This research consists of three main objectives, which are:

1. To extend the ordinary logistic equation to the fractional logistic equation.
2. To solve the fractional modified logistic equation for microalgae growth by using the Adam's-type predictor-corrector method.
3. To propose a closed-form solution for the fractional logistic equation by using Q-modified Eulerian numbers.

1.4 Research scope

In this research, the logistic equation was extended to the fractional-order model by using the fractional differential equation in the sense of Caputo derivative of $\beta \in (0,1]$. There were two versions of the fractional-order logistic equation involved in this research, addressed as the fractional modified logistic equation and the fractional logistic equation. For the modified logistic equation of microalgae growth, an investigation on the fractionalisation of the model was done by applying the optimisation approach to the obtained experiment data. The fractional modified logistic equation was then numerically solved by using the Adam's-type predictor-corrector method. The applied method was motivated by the research work by Diethelm et al. (2002). The proposed algorithm was rewritten in the form that fitted the fractional modified logistic equation. The obtained solutions with different orders of β were presented and observed.

Whereas, for the fractional logistic equation, which was motivated by the study by D'Ovidio and Loreti (2018), a closed-form solution for the equation was proposed. The proposed solution was obtained by applying Q-modified Eulerian numbers, which is believed to be a more generalised solution that achieves numerical convergence. The convergency of the proposed solution by the Q-modified Eulerian numbers was also provided. In this research, Maple 18 was used as a computational platform for both types of equations.

1.5 Significance of research

Previously, the integer-order logistic equation was used to describe population growth and other relevant physical problems. Recently, along with the advancements in calculus, it has been seen that some fractional-order equations explain certain complex physical phenomena in a better way. The development of the fractional-order equation provides opportunities to improve the application of the logistic equation, since the logistic equation serves as a method for forecasting population and studying growth patterns to aid in the decision making and future planning.

This research incorporated the application of the fractional modified logistic equation in predicting microalgae growth. In addition, a new approach for solving the logistic fractional equation was proposed. The approach provided a closed-form solution for the fractional logistic equation as the alternative for the unsolved exact solution for the fractional logistic equation. The closed-form solution is also important, as numerical computations may involve a high computational cost. Based on the conducted numerical experiment, the method was considered to be highly efficient and accurate, which would enable the method to be implemented in the future.

1.6 Framework of research

This research is organised into six chapters. The first chapter begins with an introduction to the research background. It then describes the problem statement, research objectives, research scope, significance of research and framework of research.

Chapter 2 discusses the literature on logistic equations, fractional calculus, the mathematical model in population dynamic and the numerical method in solving the mathematical model of fractional logistic equation . Chapter 2 also discusses the main research lines on logistic equations and fractional calculus, as well as describing both topics.

Chapter 3 describes the research methodology used to solve the fractional logistic equations involved in this research. It outlines the numerical approach used in solving each version of the fractional-order logistic equation.

Chapter 4 presents the results obtained by applying the fractional modified logistic equation to the microalgae growth's experiment data. The results obtained from different order of β were presented and observed.

Chapter 5 reports the results obtained from solving the fractional logistic equation by the proposed Q-modified Eulerian numbers. The connection of the proposed Eulerian numbers was described. This chapter also presents the convergency analysis results of the proposed solution.

Chapter 6 concludes the research by discussing the research results and provides recommendations to further the research in fractional calculus area especially. The framework of this research is summarised in Figure 1.1.

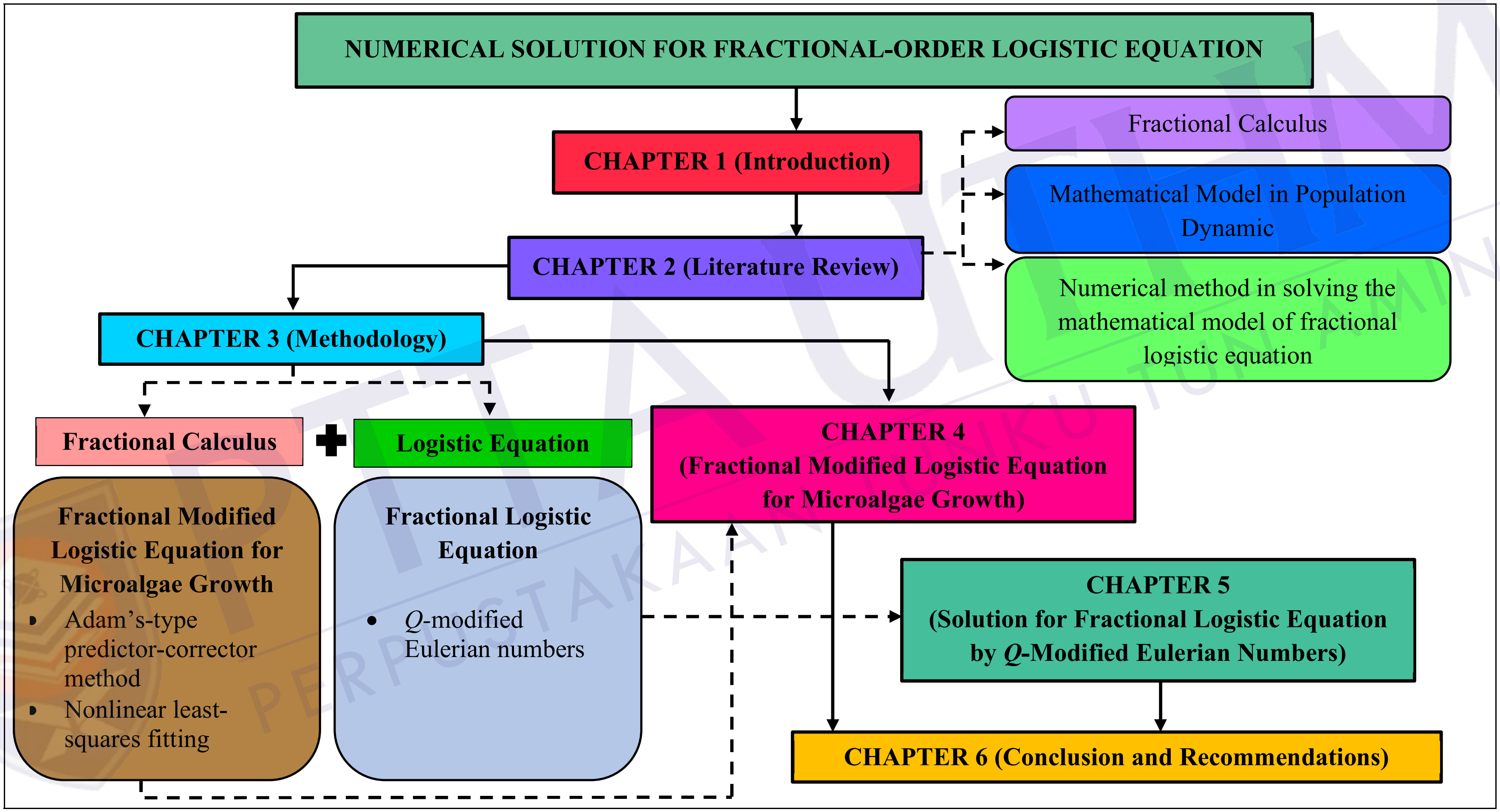

Figure 1.1: Framework of research

CHAPTER 2

LITERATURE REVIEW

2.1 Fractional calculus

The history of fractional calculus began when French mathematician L'Hopital wrote a letter to Leibniz, a fellow mathematician, on September 30th, 1695 (Podlubny, 1999). In the letter, Leibniz was asked about a particular notation that he had used in his publications for the n-th derivative of the linear function. L'Hopital queried Leibniz on what would the result of the function be if $n = 1/2$. From this exchange, fractional calculus was born. Fractional calculus is the generalisation of the classical order calculus to the non-integer-order calculus.

Just like the classical order calculus, the fractional calculus also consists of integrations and derivatives. Based on the meaning of the fractional calculus, the fractional integrals and derivatives refer to the generalisation of the classical order of one integral and derivatives to the non-integer order. Different from the classical order of one integral and derivatives, the fractional-order integrals and derivatives have their definition operators. Commonly, the fractional integral formulation can be directly derived from the classical expression of the repeated integration of a function, where this approach is called the Riemann-Liouville approach. This approach has contributed much to the theory of fractional derivatives. The derivation of the Riemann-Liouville fractional integral's definition will be presented later in this research. There are many types of operators that can be used to describe fractional derivatives but the most commonly used operators are the Riemann-Liouville and Caputo fractional derivative operators. It can be said that fractional calculus is not really a new topic of interest in mathematics, since the theory of fractional-order derivatives was developed more than three centuries ago. However, the works in solving fractional-order equations and also their applications in describing real phenomena are still undergoing massive development.

In deriving the fractional-order derivative, the idea can be seen in a simple example from the standard derivative of the monomial. The notation begins with:

$$y = f(x) = x^k$$

$$f'(x) = kx^{k-1}$$

$$f''(x) = k(k-1)x^{k-2}$$

$$f'''(x) = k(k-1)(k-2)x^{k-3}$$

.

.

.

Hence, it is concluded that

$$f^{(n)}(x) = \frac{k!}{(k-n)!}x^{(k-n)}. \tag{2.1}$$

However, this function is only applicable to the classical integer order. Hence, in generalising the function to be applicable for all real numbers, a special function called the gamma function is needed. The description of the gamma function is presented in the following subsection. In the past few decades, the interest in applying the fractional-order equation to describe some real-life phenomena was greatly developed. It was shown that some fractional-order equations describe some complex physical phenomena in a better way (Atangana & Secer, 2013). One of the most important advantages of fractional-order models in comparison with the integer-order ones is that fractional integrals and derivatives are a powerful tool for the description of memory and hereditary properties of some materials (Area, Losada, & Nieto, 2016). Furthermore, many authors pointed out that fractional-order derivatives and integrals are very appropriate for applications in various fields. One of the applications of fractional calculus was to solve the problem regarding the ultrasonic wave propagation in the human cancellous bone for early clinical detection of the osteoporosis diseases (Sebaa et al., 2006). Fractional calculus was also applied to describe the viscous interactions between fluid and a solid structure. The other application of fractional calculus was in modelling the cardiac tissue electrode interface (Magin & Ovadia, 2008). The application of fractional calculus in modelling the cardiac tissue electrode interface managed to provide an improved description of the observed bioelectrode behaviour. Apart from that, fractional calculus was also applied in the differentiation of edge detection, where it was involved in the image processing procedure (Mathieu et al., 2003). The common integer-order differentiation operators of edge detection were changed to the fractional-order type. The researchers demonstrated that introducing an edge detector based on fractional differentiation improved image processing.

Due to the greatly developing interests in the study of fractional-order equation, a considerable amount of research regarding the exact and numerical solutions of various kinds of fractional-order equations have been proposed. One of the equations is the logistic equation. The logistic equation was originally introduced by Pierre Francois Verhulst in 1838, who studied this equation in relation to population growth (Cushing, 1998). The fractional Schrödinger equation, which was discovered by Laskin (2000), is said to be a fundamental equation in fractional quantum mechanics. Other examples of the fractional-order equation are the fractional Riccati differential equation (Odibat & Momani, 2006), fractional-order Rössler equation (Li & Chen, 2004a) and space fractional Fokker Planck equation (Liu, Anh, & Turner, 2004).

2.1.1 Gamma function

Essentially, the gamma function is tied to the fractional calculus. According to Podlubny (1999), the gamma function generalises the factorial function and also is allowed to be non-integers and even complex values. The description of the gamma function is given by

$$\Gamma(z) = \int x^{z-1} e^{-x} dx \int_0^\infty x^{z-1} e^{-x} dx. \tag{2.2}$$

By applying the integration by parts, the relationship between the gamma functions (2.3), (2.4) and (2.5) can be obtained

$$\Gamma(n+1) = n\Gamma(n) \tag{2.3}$$

$$\Gamma(n) = (n-1)!, \tag{2.4}$$

REFERENCES

Abobakr, A., Said, L. A., Madian, A. H., Elwakil, A. S., & Radwan, A. G. (2017). Experimental comparison of integer / fractional-order electrical models of plant. *International Journal of Electronics and Communications(AEÜ)*, *80*, 1–9.

Alias, N. A. (2015). *The Phycoremediation of Domestic and Palm Oil Wastewater by Using Microalgae (Botryococcus sp.)*. Universiti Tun Hussein Onn Malaysia.

Allee, W. C., & Bowen, E. S. (1932). Studies in animal aggregations: mass protection against colloidal silver among goldfishes. *Journal of Experimental Zoology*, *61*(2), 185–207.

Appell, P. (1880). Sur une classe de polynômes. In *Annales Scientifiques de l'École Normale Supérieure* (Vol. 9, pp. 119–144).

Area, I., Losada, J., & Nieto, J. J. (2016). A note on the fractional logistic equation. *Physica A*, *444*, 182–187.

Atangana, A., & Kilicman, A. (2013). Analytical Solutions of the Space-Time Fractional Derivative of Advection Dispersion Equation. *Mathematical Problems in Engineering*, *2013*(1).

Atangana, A., & Secer, A. (2013). A note on fractional order derivatives and table of fractional derivatives of some special functions. *Abstract and Applied Analysis*, *2013*(1), 1–9.

Balzotti, C., D'Ovidio, M., & Loreti, P. (2020). Fractional SIS Epidemic Models. *Fractal and Fractional*, *4*(3), 44.

Bas, E., & Ozarslan, R. (2018). Real world applications of fractional models by Atangana–Baleanu fractional derivative. *Chaos, Solitons & Fractals*, *116*, 121–125.

Benzekry, S., Lamont, C., Beheshti, A., Tracz, A., Ebos, J. M. L., Hlatky, L., & Hahnfeldt, P. (2014). Classical Mathematical Models for Description and Prediction of Experimental Tumor Growth. *PLoS Computational Biology*, *10*(8).

Beverton, R. J. H., & Holt, S. J. (2012). *On the dynamics of exploited fish populations* (Vol. 11). Springer Science & Business Media.

Bhalekar, S., & Daftardar-gejji, V. (2012). Solving Fractional-Order Logistic Equation Using a New Iterative Method. *International Journal of Differential Equations*, *2012*.

Camargo, R. D. F., Gomes, A. V., & Varalta, N. (2014). A Prelude to the Fractional Calculus Applied to Tumor Dynamic. *TEMA (São Carlos)*, *15*(2), 211–221.

Caponetto, R. (2010). *Fractional order systems: modeling and control applications* (Vol. 72). World Scientific.

Caputo, M., & Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl*, *1*(2), 1–13.

Carlitz, Leo. (1959). Eulerian numbers and polynomials. *Mathematics Magazine*, *32*(5), 247–260.

Carlitz, Leonard, Kurtz, D. C., Scoville, R., & Stackelberg, O. P. (1972). Asymptotic properties of Eulerian numbers. *Probability Theory and Related Fields*, *23*(1), 47–54.

Chen, Y., Jackson, D. A., & Harvey, H. H. (1992). A comparison of von Bertalanffy and polynomial functions in modelling fish growth data. *Canadian Journal of Fisheries and Aquatic Sciences*, *49*(6), 1228–1235.

Costabile, F. A., & Longo, E. (2012). Algebraic theory of Appell polynomials with application to general linear interpolation problem. *Linear Algebra-Theorems and Applications. InTech, Croatia*, 21–46.

Cushing, J. M. (1998). *An introduction to structured population dynamics*. *Society for Industrial and Applied Mathematics (SIAM)* (Vol. 71).

Das, S., & Gupta, P. K. (2011). A mathematical model on fractional Lotka – Volterra equations. *Journal of Theoretical Biology*, *277*(1), 1–6.

Davison, M., & Essex, C. (1998). Fractional differential equations and initial value problems. *Mathematical Scientist*, *23*(2), 108–116.

Demirci, E., & Ozalp, N. (2012). A method for solving differential equations of fractional order. *Journal of Computational and Applied Mathematics*, *236*(11), 2754–2762.

Dhaka, P., & Singh, G. P. (2016). Effect of salinity on growth and biopigment composition of dunaliella salina isolated from sambhar salt lake, Rajasthan,(india). *Int J Pharm Bio Sci*, *7*(1), 528–533.

Diethelm, K. A. I., Ford, N. J., & Freed, A. D. (2002). A Predictor-Corrector Approach for the Numerical Solution of Fractional Differential Equations. *Nonlinear Dynamics*, *29*, 3–22.

Diethelm, K., Ford, N. D. J., & Freed, A. D. (2001). A Predictor Corrector Approach For the Numerical Solution of Fractional Differential Equations.

Diethelm, K., & Ford, N. J. (2002). Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications*, *265*(2), 229–248.

Du, M., Wang, Z., & Hu, H. (2013). Measuring memory with the order of fractional derivative, (1), 1–3.

Dvoretsky, D., Dvoretsky, S., Peshkova, E., & Temnov, M. (2015). Optimization of the process of cultivation of microalgae chlorella vulgaris biomass with high lipid content for biofuel production. *CHEMICAL ENGINEERING*, *43*.

Dzieli, A., Sarwas, G., & Sierociuk, D. (2011). Comparison and validation of integer and fractional order ultracapacitor models. *Advances in Difference Equations*, 1–15.

El-sayed, A. M. A., El-mesiry, A. E. M., & El-saka, H. A. A. (2007). On the fractional-order logistic equation. *Applied Mathematics Letter*, *20*, 817–823.

Euler, L. (1749). Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques, Académie des sciences de Berlin, Lu en 1749. *Opera Omnia Serie I*, *15*(1749), 70–90.

Ezz-Eldien, S. S. (2018). On solving fractional logistic population models with applications. *Computational and Applied Mathematics*, *37*(5), 6392–6409.

Figueiredo Camargo, R., Chiacchio, A. O., & Capelas de Oliveira, E. (2008). Differentiation to fractional orders and the fractional telegraph equation. *Journal of Mathematical Physics*, *49*(3), 33505.

Foata, D. (2010). Eulerian polynomials: from Euler's time to the present. In *The legacy of Alladi Ramakrishnan in the mathematical sciences* (pp. 253–273). Springer.

Freeborn, T. J. (2013). A Survey of Fractional-Order Circuit Models for Biology and Biomedicine. *Journal On Emerging And Selected Topic In Circuits And Systems*, *3*(3), 416–424.

Gani, P., Mohamed Sunar, N., Matias-Peralta, H., Abdul Latiff, A. A., & Mohamad Fuzi, S. F. Z. (2017). Growth of microalgae Botryococcus sp. in domestic wastewater and application of statistical analysis for the optimization of flocculation using alum and chitosan. *Preparative Biochemistry and*

Biotechnology, *47*(4), 333–341.

Gompertz, B. (1825). XXIV. On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. In a letter to Francis Baily, Esq. FRS &c. *Philosophical Transactions of the Royal Society of London*, (115), 513–583.

Grünwald, A. K. (1867). Über "begrenzte" derivationen und deren anwendung. *Zeitschrift F¨ur Mathematik Und Physik*, *12*, 441–480.

He, J.-H. (1999). Variational iteration method–a kind of non-linear analytical technique: some examples. *International Journal of Non-Linear Mechanics*, *34*(4), 699–708.

I. Podlubny. (1999). *Fractional Differential Equations*. New York: Academic Press.

Iijima, K., Sasaki, K., Takahashi, Y., & Yoshinaga, M. (2016). Eulerian polynomials and polynomial congruences. *ArXiv Preprint ArXiv:1611.10147*.

Isah, A., Phang, C., & Phang, P. (2017). Collocation method based on Genocchi operational matrix for solving generalized fractional pantograph equations. *International Journal of Differential Equations*, *2017*.

Jalalizadeh, M. (2012). Development of an integrated process model for algae growth in a photobioreactor.

Jamaian, S. S., Bakeri, N. M., Sunar, N. M., & Gani, P. (2017). A Verhulst model for microalgae Botryococcus sp. growth and nutrient removal in wastewater. In *AIP Conference Proceedings* (Vol. 1870, p. 40054). AIP Publishing LLC.

Jin, B., Lazarov, R., & Zhou, Z. (2015). An analysis of the L1 scheme for the subdiffusion equation with nonsmooth data. *IMA Journal of Numerical Analysis*, *36*(1), 197–221.

Jumarie, G. (2005). On the solution of the stochastic differential equation of exponential growth driven by fractional Brownian motion. *Applied Mathematics Letters*, *18*(7), 817–826.

Jumarie, G. (2012). An approach to differential geometry of fractional order via modified Riemann-Liouville derivative. *Acta Mathematica Sinica, English Series*, *28*(9), 1741–1768.

Jumarie, G. (2013). On the derivative chain-rules in fractional calculus via fractional difference and their application to systems modelling. *Open Physics*, *11*(6), 617–633.

Katsanevakis, S., & Maravelias, C. D. (2008). Modelling fish growth: multi-model inference as a better alternative to a priori using von Bertalanffy equation. *Fish and Fisheries*, *9*(2), 178–187.

Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier B.V.

Kingsland, S. E. (1985). Modeling nature. *University of Chicago Press*.

Laird, A. K. (1965). Dynamics of tumour growth: comparison of growth rates and extrapolation of growth curve to one cell. *British Journal of Cancer*, *19*(2), 278.

Laskin, N. (2000). Fractional quantum mechanics. *Physical Review E*, *62*(3), 3135–3145.

Lester, N. P., Shuter, B. J., & Abrams, P. A. (2004). Interpreting the von Bertalanffy model of somatic growth in fishes: the cost of reproduction. *Proceedings of the Royal Society of London. Series B: Biological Sciences*, *271*(1548), 1625–1631.

Letnikov, A. V. (1868). Theory of Differentiation with an Arbtraly Indicator. *Matem Sbornik*, *3*, 1–68.

Li, Changpin, Qian, D., & Chen, Y. (2011). On Riemann-Liouville and Caputo derivatives. *Discrete Dynamics in Nature and Society*, *2011*.

Li, Chunguang, & Chen, G. (2004a). Chaos and hyperchaos in the fractional-order Rössler equations. *Physica A: Statistical Mechanics and Its Applications*, *341*(1–4), 55–61.

Li, Chunguang, & Chen, G. (2004b). Chaos in the fractional order Chen system and its control, *22*, 549–554.

Liu, F., Anh, V., & Turner, I. (2004). Numerical solution of the space fractional Fokker – Planck equation, *166*, 209–219.

Loh, J. R., & Phang, C. (2019). Numerical solution of Fredholm fractional integro-differential equation with right-sided Caputo's derivative using Bernoulli polynomials operational matrix of fractional derivative. *Mediterranean Journal of Mathematics*, *16*(2), 1–25.

Lotka, A. J. (1925). Elements of physical biology. Williams and Wilkins. *Baltimore, Md*.

Lu, J. G., & Chen, G. (2006). A note on the fractional-order Chen system, *27*, 685–688.

Lucas, W., & Thomas J, O. (2006). Translation with notes of Euler's paper Remarques sur un beau rapport entre les series des puissances tant directes que reciproques.

M.Caputo. (1967). Linear models of dissipation whose Q is almost frequency independent. *Journal of the Royal Astronomical Society of Canada*, *529–539*.

Machado, J. T., Kiryakova, V., & Mainardi, F. (2011). Recent history of fractional calculus. *Communications in Nonlinear Science and Numerical Simulation*, *16*(3), 1140–1153.

Magin, R. L., & Ovadia, M. (2008). Modeling the cardiac tissue electrode interface using fractional calculus. *Journal of Vibration and Control*, *14*(9–10), 1431–1442.

Mata, T. M., Martins, A. A., & Caetano, N. S. (2010). Microalgae for biodiesel production and other applications: a review. *Renewable and Sustainable Energy Reviews*, *14*(1), 217–232.

Mathieu, B., Melchior, P., Oustaloup, A., & Ceyral, C. (2003). Fractional differentiation for edge detection. *Signal Processing*, *83*(11), 2421–2432.

Miller, K. S., & Ross, B. (1993). *An Intro to the Fractional Calculus and Fractional Differential Equations*. New York: A Willey- Interscience Publication.

Mohamed, M. S., & Sherif, E.-B. (2013). Optimal Homotopy Analysis Method to Solve the Fractional Order Logistic Equation, *1*(1), 1–9.

Monje, C. A., Chen, Y., Vinagre, B. M., Xue, D., & Feliu-Batlle, V. (2010). *Fractional-order systems and controls: fundamentals and applications*. Springer Science & Business Media.

Muniswamy, S. (2014). *Analytical and Numerical Approach to Caputo Fractional Differential Equations via Generalized Iterative Schemes with Applications*.

N.H., Sweilam;M.M, Khader;A.M.S, M. (2012). Numerical Studies For Solving Fractional-Order Logistic Equation, *78*(8), 1199–1210.

Nkiruka.C, A., John.C, G., & Thomas.E, Y. (2010). The integration of quantitative multi-modality imaging data into mathematical models of tumors. *National Institute of Health*, *29*(10), 1883–1889.

Norton, L. (1988). A Gompertzian model of human breast cancer growth. *Cancer Research*, *48*(24 Part 1), 7067–7071.

Noupoue, Y. Y. Y., Tandoğdu, Y., & Awadalla, M. (2019). On numerical techniques for solving the fractional logistic differential equation. *Advances in Difference Equations*, *2019*(1), 108.

Odibat, Z., & Momani, S. (2006). Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order. *Chaos, Solitons and Fractals*, *36*(1), 167–174.

Odibat, Z., & Momani, S. M. (2008). An Algorithm for the numerical solution of differential equations of fractional order. *Journal of Applied Mathematics and Informatics*, (December 2014).

Ortigueira, M., & Bengochea, G. (2017). A new look at the fractionalization of the logistic equation. *Physica A: Statistical Mechanics and Its Applications*, *467*, 554–561.

Ovidio, M. D., & Loreti, P. (2018). Solutions of fractional logistic equations by Euler's numbers. *Physica A*.

Pielou, E. C. (1974). *Population and community ecology: principles and methods*. CRC Press.

Priyadarshani, I., & Rath, B. (2012). Commercial and industrial applications of micro algae–A review. *J Algal Biomass Utln*, *3*(4), 89–100.

Rabiei, K., Ordokhani, Y., & Babolian, E. (2018). Numerical Solution of 1D and 2D fractional optimal control of system via Bernoulli polynomials. *International Journal of Applied and Computational Mathematics*, *4*(1), 7.

Ramos, R. A. (2013). Logistic function as a forecasting model: It's application to business and economics. *International Journal of Engineering*, *2*(3), 2305–8269.

Riemann, B. (1876). Versuch einer allgemeinen Auffassung der Integration und Differentiation. *Gesammelte Werke*, *62*(1876).

Riesz, M. (1949). L'intégrale de Riemann-Liouville et le problème de Cauchy. *Acta Mathematica*, *81*(1), 1–222.

Riesz, M. M. (1939). L'integrale De Riemann-Liouville et le problème de Cauchy pour l'´equation des ondes. *Bulletin de La SMF*, *67*, 153–170.

Rodabazlgh, D., Wesson, J. R., Marshall, G. C., & Flight, S. (1965). On the Efficient Use of Predictor-Corrector Methods In the Numerical Solution of Differential Equations.

Ruiz, J., Arbib, Z., Álvarez-Díaz, P. D., Garrido-Pérez, C., Barragán, J., & Perales, J. A. (2013). Photobiotreatment model (PhBT): a kinetic model for microalgae biomass growth and nutrient removal in wastewater. *Environmental Technology*, *34*(8), 979–991.

Sarawak Biodiversity Centre. (2019). Botryococcus sp. Retrieved from https://www.sbc.org.my/programmes/r-d-bioprospecting/algae-research/microalgae-collection/187-english-categories/programmes/r-d-laboratories/algae-research/featured-algae/860-botryococcus-braunii

Sebaa, N., Fellah, Z. E. A., Lauriks, W., & Depollier, C. (2006). Application of fractional calculus to ultrasonic wave propagation in human cancellous bone. *Signal Processing*, *86*(10), 2668–2677.

Spolaore, P., Joannis-Cassan, C., Duran, E., & Isambert, A. (2006). Commercial applications of microalgae. *Journal of Bioscience and Bioengineering*, *101*(2), 87–96.

Spratt, J. A., Von Fournier, D., Spratt, J. S., & Weber, E. E. (1993). Decelerating growth and human breast cancer. *Cancer*, *71*(6), 2013–2019.

Sweilam, N., & Nagy, A. M. (2019). Nonstandard Finite Difference Scheme For The Fractional Order Salmonella Transmission Model, *10*(1).

Sweilam, N H, Khader, M. M., & Al-Bar, R. F. (2007). Numerical studies for a multi-order fractional differential equation. *Physics Letters A*, *371*(1–2), 26–33.

Sweilam, Nasser Hassan, Khader, M. M., & Mahdy, A. M. S. (2012). Numerical studies for fractional-order Logistic differential equation with two different delays. *Journal of Applied Mathematics*, *2012*.

Thurman, H. V, & Burton, E. A. (1997). *Introductory oceanography*. Prentice Hall New York.

Ul, Z., Zafar, A., Mushtaq, M., & Rehan, K. (2018). A non-integer order dengue internal transmission model. *Adv Differ Equ*. https://doi.org/10.1186/s13662-018-1472-7

Ummal Aisha Farhana, M. S., Meow En, Y., & Siti Suhana, J. (2018). The Extended Monod Model for Microalgae Growth and Nutrient Uptake in Different Wastewaters. *International Journal of Engineering and Technology*, *7*, 200–204.

Usmani, M., & Sultana, H. (2014). Use of microalgae for the removal of environmental pollutants. *International Journal of Scientific World*, *2*.

Valentim Jr, C. A., Oliveira, N. A., Rabi, J. A., & David, S. A. (2020). Can fractional calculus help improve tumor growth models? *Journal of Computational and Applied Mathematics*, 112964.

Valentina, S. (2015). Logistic Equation and its Application as Forecasting Model of Vegetables Production in Greenhouses in Albania, (October).

Vandermeer, J. H., & Goldberg, D. E. (2013). *Population ecology: first principles*. Princeton University Press.

Vivek, D., Mission, R., & Vivek, D. (2016). Numerical solution of fractional-order logistic equations by fractional Euler ' s method. *International Journal for Research in Applied Science & Engineering Technology (IJRASET)*, (March).

Von Bertalanffy, L. (1938). A quantitative theory of organic growth (inquiries on growth laws. II). *Human Biology*, *10*(2), 181–213.

Weedon-Fekjær, H., Lindqvist, B. H., Vatten, L. J., Aalen, O. O., & Tretli, S. (2008). Breast cancer tumor growth estimated through mammography screening data. *Breast Cancer Research*, *10*(3), 1–13. https://doi.org/10.1186/bcr2092

West, B. J. (2015). Exact solution to fractional logistic equation. *Physica A: Statistical Mechanics and Its Applications*, *429*, 103–108.

Weyl, H. (1917). Bemerkungen zum begriff des differentialquotienten gebrochener ordnung. *Zürich. Naturf. Ges*, *62*, 296–302.

Winsor, C. P. (1932). The Gompertz curve as a growth curve. *Proceedings of the National Academy of Sciences of the United States of America*, *18*(1), 1.

Wood, D C. (1992). *The computation of polylogarithms. University of Kent computing laboratory*. technical report 15/92.

Wood, David C. (1992). The computation of polylogarithms.

Yan, Y., Khan, M., & Ford, N. J. (2018). An analysis of the modified L1 scheme for time-fractional partial differential equations with nonsmooth data. *SIAM Journal on Numerical Analysis*, *56*(1), 210–227.

Yong Zhou; JinRong Wang; Lu Zhang. (2017). *Basic Theory of Fractional Differential Equations*.

Zullaikah, S., Utomo, A. T., Yasmin, M., Ong, L. K., & Ju, Y. H. (2019). Ecofuel conversion technology of inedible lipid feedstocks to renewable fuel. In *Advances in Eco-Fuels for a Sustainable Environment* (pp. 237–276). Elsevier.