SOLVING THE VOLTERRA INTEGRAL FORM OF THE LANE EMDEN EQUATIONS BY HOMOTOPY PERTURBATION METHOD AND ADOMIAN DECOMPOSTION METHOD

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DEDICATION

To my beloved Father and Mother

Baharom Bin Safon & Fauziah Binti Md Yatim

My siblings for the help

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And my friend that always help in finishing the writing

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ABSTRACT

In this research, Homotopy Perturbation Method(HPM) and Adomian Decomposition Method(ADM), as two known methods, is analyzed for solving Volterra integral from of the Lane-Emden equations. The result from both method are compared with the exact solution. It has been shown that these two methods are capable to get the similar result with exact solution, when solving Volterra integral from of the Lane-Emden equations. The efficiency of these two methods are demonstrated through several examples to show the ability of the methods in solving Lane-Emden equations of Volterra integral form. This research has determind that ADM is more reliable and efficient than HPM. The ADM requires shorter computational that HPM, but it yields more accurate results compared to HPM.

ABSTRAK

Dalam kajian ini, kaedah pengusikan Homotopy (HPM) dan kaedah Penguraian Adomian(ADM) sebagai dua kaedah yang diketahui, dianalisis untu menyelesaikan pengamiran Volterra dari persamaan Lane-Emden. Hasil daripada kedua-dua kaedah ini dibandingkan dengan penyelesaian yang tepat. Telah ditunjukkan bahawa kedua-dua kaedah ini mampu menghasilkan hasil yang sama dengan penyelesaian yang yang tepat apabila menyelesaikan pengamiran Volterra dari persamaan Lane-Emden. Kecekapan kedua-dua kaedah ini ditunjukkan melalui beberapa contoh auntuk menunjukkan keupayaan kaedah dalam menyelesaiakan persamaan Lane-Emden bentuk Volterra. Kajian ini telah menentukan bahawa ADM adalah lebih dipercayai dan cekap daripada HPM. ADM memerlukan pengiraan yang lebih pendek darpada HPM tetapi menhasilkan hasil yang lebit tepat berbanding HPM.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS AND ABBREVIATIONS

ADM - Adomian Decomposition Method

HPM- Homotopy Perturbation Method

LIST OF APPENDIX

CHAPTER 1

INTRODUCTION

1.1 Research Background

Several scientific and engineering applications are usually described by integral equations or integro differential equations, standard or singular. There are large class of initial and boundary value problem that can be converted to Volterra or Fredholm integral equations. Integral equations arise in the potential theory more than any other field. Integral equations arise also in diffraction problems, conformal mapping, water waves, scattering in quantum mechanics, and Volterra's population growth model. The electrostatic, electromagnetic scattering problems and propagation of acoustical and elastical waves are scientific fields where integral equations appear.

1.1.1 Volterra In tegral From Of The Lane-Emden Equation

The Lane-Emden equation appears mostly in astrophysics, such as the theory of stellar structure, and the thermal behavior of a spherical cloud of gas. The Lane-Emden equation comes in two kinds, namely the Lane-Emden equations (Wazwaz et al. 2013) of the first kind, or of index m, that reads

$$y'' + \frac{k}{x}y' + y^m = 0\,, y(0) = 1\,, y'(0) = 0\,, k \geq 1\,, \tag{1.1}$$

and the Lane-Emden equation of the second kind in the form

$$y'' + \frac{k}{x}y' + e^{y} = 0\,, y(0) = y'(0) = 0\,, k \geq 1\,, \tag{1.2}$$

The Lane-Emden equation of the first kind appears in astrophysics and used for computing the structure of interiors of polytropic stars, where exact solutions exist for $m = 0, 1, 5$. However, the Lane-Emden equation of the second kind models the non-dimensional density distribution $y(x)$ in an isothermal gas sphere. The singular behavior of these equations that occurs at $x = 0$ is the main difficulty of the singular behavior.

The generalized Lane-Emden equation of the shape factor of $k > 1$ reads

$$y'' + \frac{k}{x}y' + f(y) = 0\,, y(0) = \alpha, y'(0) = 0\,, k > 1\,, \tag{1.3}$$

where $f(y)$ can take y^{m} or $e^{y(x)}$ as given earlier. To convert (1.3) to an integral form, it was given that

$$y(x) = \alpha - \frac{1}{k-1}\int_{0}^{x} t\left(1 - \frac{t^{k-1}}{x^{k-1}}\right) f\big(y(t)\big)dt \tag{1.4}$$

Differentiating (1.4) twice, using the Leibniz's rule, gives

$$y'(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f\big(y(t)\big)\, dt\,,$$

$$y''(x) = -f\big(y(x)\big) + \int_0^x k\left(\frac{t^k}{x^{k+1}}\right) f\big(y(t)\big) dt \tag{1.5}$$

that can be proved by multiplying y' by $\frac{k}{x}$ and adding the result to $y''(x)$. This shows that the Lane-Emden equation is equivalent to the Volterra integral from (1.4) or to the two Volterra integro-differential forms in (1.5) of first-order and second-order respectively.

It is to be noted that equations (1.4) and (1.5) work for any function $f(y)$, such as y^m and e^y as in the first kind and the second kind of the Lane-Emden equation. Moreover, $f(y)$ can be linear or nonlinear function of y. A significant feature of equations (1.4) and (1.5) is the overcome of the singular behavior at $x = 0$.

Moreover, for $k = 1$ the integral form is

$$y(x) = \alpha + \int_0^x t \ln\left(\frac{t}{x}\right) f\big(y(t)\big) dt\,, \tag{1.6}$$

which can be obtained by setting $k \to 1$ in Eq (1.4).

Based on the last results we set the Volterra integral forms for the Lane-Emden equations as

$$y(x) = \begin{cases} \alpha + \int_0^x t \ln\left(\frac{t}{x}\right) f\big(y(t)\big) dt & for\ k \\ \alpha - \frac{1}{k-1}\int_0^x t\left(1 - \frac{t^{k-1}}{x^{k-1}}\right) f\big(y(t)\big) dt & for\ k > 1 \end{cases} \tag{1.7}$$

1.1.2 Homotopy Perturbation Method And Adomian Decomposition Method

In recent years, a growing development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical techniques for nonlinear problems. The Homotopy Perturbation Method (HPM) is a powerful and efficient technique to find the solutions of linear and nonlinear equations. HPM is a combination of the perturbation and homotopy methods.

A homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times$

$[0,1] \rightarrow Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that, if $x \in X$ then $H(x,0) = f(x)$ and $H(x,1) = g(x)$. The second parameter of H as time then H describes a continuous deformation of f into g, at time 0 will have the function f and at time 1 will have the function g.

Perturbation theory is applicable if the problem at hand cannot be solved exactly, but can be formulated by adding a “small” term to the mathematical description of the exactly solvable problem. Perturbation theory leads to an expression for the desired solution in terms of a formal power series in some “small” parameter which known as a perturbation series, that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem. Formally, the approximation to the full solution A, a series in the small parameter, like the following:

$$A = A_0 + PA_1 + P^2A_2 + \cdots \tag{1.8}$$

In this example, A_0 would be the known solution to the exactly solvable initial problem and A_1, $A_2, \cdots$ represent the higher-order terms which may be found iteratively by some systematic procedure. For small p these higher-order terms in the series become successively smaller.

The HPM is a series expansion method used in the solution of nonlinear partial differential equations. The method employs a Homotopy transform to generate a convergent series solution of differential equations. This gives flexibility in the choice of basis functions for the solution and the linear inversion operators, while still retaining a simplicity that makes the method easily understandable from the standpoint of general perturbation methods. The HPM proposed first by the present author in 1998, was further developed and improved by He. He is the founder of HPM and he proved that HPM can solve various kinds of nonlinear functional equations because this method yields a very rapid convergence of the solution series in most cases. Besides that only a few iterations leading to very accurate solutions. Recently, many authors applied this method to various problems and demonstrated the efficiency of it to handle nonlinear structures and solve various physics and engineering problems.

At the beginning of the 1980's, an American Mathematician named George Adomian presented a powerful decomposition methodology for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations (ODE), partial differential equations (PDE), integral equations, etc. Since then, the method has been known as the Adomian decomposition method or in short the ADM. The ADM is a significant, powerful method, which provides an efficient means for the analytical and numerical solution of differential equations, which model real-world physical applications.

1.2 Problem Statement

Recently, there are lot of model consists of an algebraic equation, integral equations, or ordinary, partial of differential equation. HPM and ADM are two of the methods to solve various kind of problem.

It has previously been observed that HPM is the efficiency of the method. Many researchers found that the HPM requires less computational work and these method is able to solve nonlinear problems without linearization. In contrast, HPM method do have some disadvantages which is HPM method gives a series solution which must be truncated for practical applications.

In other hand, ADM method shows that the rate of region of convergence are potential shortcoming which is the series will rapid convergent in a very small region but it has very slow convergence rate in the wider region and the truncated series solution is an inaccurate solution in that region.

1.3 Objective

The aim of this study is to determine the efficiency of HPM and ADM for solving the Volterra integral from of the Lane-Emden equations. Therefore, the objective is to ensure that the research done on the track and in the right perspective.

The specific goal and objectives of this study are:

i) To study the method of HPM and ADM for solving Lane-Emden equation of Volterra integral form.
ii) To determine the accuracy and efficient of HPM and ADM in solving Lane-Emden equation by compare the solution of both method with exact solution.
iii) To compare the rate of convergence between HPM and ADM by using the tabulated data and illustrated by graph.

1.4 Scope of Study

This study will focus on solving Lame-Emden equation by using HPM and ADM in linear or non-linear ODE. Severe example of Lane-Emden equation will be tested to shows that the HPM and ADM gives the accuracy and efficient of numerical solution.

1.5 Significant of Research / Importance of Study

There is several significant of this research:

i) To solve many functional equations using HPM and ADM that require less calculations.
ii) The HPM and ADM can improve the convergence or accuracy of the series solution.

1.6 Thesis Outline

This project consists of an introductory chapter and four main chapters dealing with the following problems:

Chapter 1: Establish the purpose of the research, problem statement, the scope of the research and method to be used.

Chapter 2: Relevant literature review about the Volterra integral equations, Lane-Emden equations, HPM, ADM, applications and advantages of the methods.

Chapter 3: Present the method that will be used to solve Volterra integral from of the Lane-Emden equations by using HPM and ADM.

Chapter 4: Present the result and discussion that have been obtained from the methods that have been used.

Chapter 5: Present the conclusion of the objectives that have been achieved. Some recommendation for the future work also mentioned in this chapter.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter discusses the literature reviews related to this study. The items of discussion are about what other people have done related to solving Volterra integral from of the Lane-Emden equations by HPM and ADM. There is also a discussion on the outcome result from other journals, articles and books.

2.2 Volterra Integral Equations

As we known the Volterra integral equation had raised in many scientific applications such as the semi-conductor devices, population dynamics and spread of epidemics. Volterra integral equations, of the first kind or the second kind, are characterized by a variable upper limit of integration.

For the first kind of Volterra integral equation, (Biazar et al. 2003) had proved that linear and nonlinear Volterra integral equation of the first kind can be solved efficienly by using ADM. While, the second kind of Volterra integral equation, (Maleknejad 2007) had use orthogonal Chebshev polynomials to approximate linear and nonlinear part of the Volterra integral equation. A quadrature method is used to compute the coefficients of expansion of any given function.

Wazwaz and Rach (2016) had examined the weakly singular Volterra integral equation by using numerical methods such as the collocation method and the split interval method which are ADM and Variational iteration method(VIM) that requires the use of the Lagrange multiplier. The two methods were proved to be powerful and efficient, where each method has its own significant features. Variety of problems can be solved through convergent power series which are ADM used the forcing function $f(x)$ through $u_0(x)$ by adomian series, where the VIM used this function in every iteration process provides successive approximation that will converge to the exact solution.

Chen 2017 had obtain approximate solutions Volterra-Hammerstein integral equation by elaborated a spectral collocation method based on legendre orthogonal polynomials. The strategy is derived using some variable transformations to change the equation into another Volterrra-Hammerstein integral equation, so that the Legendre orthogonal polynomial theory can be applied conveniently. By using gauss quadrature formula , all the integral terms are approximated.

2.3 Lane-Emden Equations

Many problems arising in the field of mathematical physics and astrophysics can be modeled by Lane-Emden. (Yıldırım and Öziş, 2009) Lane-Emden with initial value problems can be written in the form:

$$y'' + \frac{2}{x}y' + f(y) = 0, \tag{2.1}$$

subject to conditions

$$y(0) = \text{A}, \qquad y'(0) = \text{B} \tag{2.2}$$

where A and B are constants and $f(y)$ is a real-valued continuous function. This equation was used to model various phenomena such as the theory of stellar structure, te thermal behavior of a spherical cloud of gas and isothermal gas spheres.

Yıldırım and Öziş (2009) had successfully employed to obtain the approximate-exact solutions of various Lane-Emden type equations by using variational iteration method. The method yields solutions in forms of convergent and proved for singular IVPs Lane-Emden type, and in some cases, yields exact solutions in few iterations.

Then Šmarda and Khan (2015) presented an efficient computational approach to solving singular initial value problems for Lane-Emden type equations with nonlinear terms based on the differential transformation method and the modified general formula for Adomian polynomials with differential transformation components. The presented method can be applied in a direct way, no need for calculating multiple integrals or derivatives and less computational work is demanded compared to other popular method.

Rismani and Monfared (2012) present an efficient numerical shceme, overcome the difficulty of singular point by applying method of variation for transforming the Lane-Emden equation to Fredholm-Volterra integral equations with smooth kernel. The Gaussian integration method with the interpolation were employed to reduce the problem to the solution of non-linear algebraic equations.

2.4 Homotopy Perturbation Method

HPM is a series expansion method and it is efficient in solving nonlinear partial differential equations. The method employs a homotopy transform to generate a convergent series solution of differential equations. This gives flexibility in the choice of basis functions for the solution and the linear inversion operators, while still retaining a simplicity that makes the method easily understandable from the standpoint of general perturbation methods.

The Homotopy Perturbation Method (HPM) was introduced and developed by Ji-Huan He (He,1999). He developed the HPM for solving nonlinear initial and boundary value problem by combining the standard homotopy in topology and the perturbation method. The standard homotopy in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a "small parameter" while the perturbation method are based on assumption that a small parameter must exist in the equation. By this method, a rapid convergent series solution can be obtained in most the cases.

He (2009) studied an elementary introduction to the concepts of HPM. There are several basic solution procedure of the HPM such as qualitative sketch/trial function solution, construction of homotopy equations, solution procedure similar to that of classical perturbation methods, optimal identification of the unknown parameter in the trial functions and higher order approximations.

Then in Demir et al. (2013) presented the mathematical analysis to acquire the solution of first-order inhomogeneous PDE through constituting the proper homotopy occupy decomposing of $f(x, y)$ was a effective tool for calculating the exact or approximate solutions with minimum computation and accelerate the convergence of the solution.

2.4.1 Application of HPM

HPM had shown to be highly accurate, and only a few terms are required to obtain accurate computable solution in solving linear and nonlinear equations. In example, this method can be applied in many applications in such fields as chemistry, physics, biology and engineering. In fact HPM can be apply to mass or heat transfer, reaction-diffusion brusselator model, thermal-diffusion, conductive-radiative fin and the wave equation.

Application of HPM with Chebyshev’s polynomials to nonlinear problems was explore by Chun (2010) who stated that differential equations can be solved by using Chebyshev’s polynomials and He’s polynomials. Chun demonstrated that the Chebyshev-based HPM shows a much better performance over the Taylor-based HPM in handling nonlinear differential equations with the not easily integrable source term.

Recently, the peristaltic motion of incompressible micropolar non-Newtonian nanofluid with heat transfer in a two-dimensional asymmetric channel is investigated under long-wavelength assumption. The flow includes radiation and viscous dissipation effects as well as all micropolar fluid parameters. The fundamental equations which govern this flow have been modeled under long- wavelength assumption, and the expressions of velocity and microrotation velocity are obtained in a closed form, while the solutions of both temperature and nanoparticles phenomena are obtained using the HPM. The different values of temperature and becomes greater with increasing the normal-axis y and reaches the maximum value at $y \approx 1.2$. (Abou-zeid 2016).

Next, the existing multivariate models in finance are based on diffusion models that lead to the need of solving systems of Riccati differential equations. The researcher combine Laplace transform and HPM is considered as an algorithm to the exact solution of the nonlinear Riccati equations , applied the technique in order to solve stiff diffusion model problems that include interest rates models as well as two and three-factor stochastic volatility models and show that the present approach is relatively easy, efficient and highly accurate. (Rodrigue et al. 2016)

Other than that, HPM is used to analyze the effects of environmental temperature and surface emissivity parameter on the temperature distribution, efficiency and heat transfer rate of a conductive– radiative fin with variable thermal conductive and surface emissitivity. HPM being one of the semi-numerical methods for highly nonlinear and inhomogeneous equations, the local temperature distribution efficiencies and heat transfer rates in which Newton–Raphson method is used for the insulated boundary condition. (Roy, Das, Mondal & Mallick, 2015).

Furthermore, HPM does not need small parameters in the equations, is compared with the perturbation and numerical methods in the heat transfer field. The perturbation method depends on small parameter assumption, and the obtained results, in most cases, end up with a non-physical result, the numerical method leads to inaccurate results when the equation is intensively dependent on time, while HPM overcomes completely the above shortcomings, revealing that the HPM is very convenient and effective. Comparing different methods show that, when the effect of the nonlinear term is negligible, homotopy perturbation method and the common perturbation method have got nearly the same answers but when the nonlinear term in the heat equation is more effective, there will be a considerable difference between the results. As the homotopy perturbation method does not need a small parameter, the answer will be nearer to the exact solution and also to the numerical one.(Ganji, 2006)

Lastly, our final example of application of this method was provided by Cuce and Cuce. Thermal performance of straight fins with both constant and temperature-dependent thermal conductivity has been investigated in detail and dimensionless analytical expressions of fin efficiency and fin effectiveness have been developed for the first time via HPM. Governing equations have been formulated by performing Darcy’s model. Dimensionless temperature distribution along the length of porous fin has been determined as a function of porosity and convection parameters. The ratio of porous fin to solid fin heat transfer rate has also been evaluated as a function of thermogeometric fin parameter. The expressions developed are beneficial for thermal engineers for preliminary assessment of thermophysical systems instead of consuming time in heat conduction problems governed by strongly nonlinear differential equations (Cuce and Cuce, 2015)

2.4.2 Advantages of HPM

A large amount of literature had showed many advantages of using Homotopy Perturbation Method. Most of the researcher states is the efficiency of the method. (Sadighi & Ganji 2007) developed HPM and ADM to obtain the exact solutions of Laplace equation. The results were compared with Variation Iteration Method. It is apparently seen that these methods are very powerful and efficient techniques for solving different kinds of problems arising in various fields of science and engineering and present a rapid convergence for the solutions. The solutions obtained show that the results of these methods are in agreement but HPM is an easy and convenient one.

Research made by (Chowdhury et al. 2010) stated advantage of HPM had twofold. Firstly, the HPM reliability and reduction in size of computational work. Secondly, in comparison with existing methods, the HPM is an improving with regard to its accuracy and rapid convergence. HPM also had advantage of being more concise for analytical and numerical purposes.

2.5 Adomian Decomposition Method

G.Adomian (Adomian,1988) was the individual that introduced and developed the ADM. This method has been applied to a wide class of linear and nonlinear Ordinary Different Equations (ODEs), Partial Differential Equations (PDEs), Integral Equations and Integro-differential Equations. can be used to obtain solutions of non-linear functional equations. ADM will give a series solution that will rapidly convergent and easily computable components.

Convergence acceleration techniques are used to accelerate convergence of a sequence or series, and even to extend the region of convergence for IVP fractional ordinary differential equations with the diagonal Pade' approximations or the iterated Shanks transforms. By using Mathematica software, this methods demonstrate that the diagonal Pade' approximations or the iterated Shanks transforms can both efficiently extend the effective convergence region of the decomposition solution (Duan et al. 2013).

2.5.1 Applications of ADM

ADM had shown that the method will provide an exact solution in solving the algebraic equations, integral equations, ordinary and partial differential equations and integro differential equations. In example, this method can be applied in many applications in such fields as chemistry, physics, biology and engineering. In fact ADM can be apply to mass or heat transfer, nonlinear optics, chaos theory, and the fermentations process.

In the Optical field, study of cases as an application for nonlinear Schrodinger equation and consequncet changing parameters of Thirring solitons for bright and drak solitons by Bakodah et al. The birefringent optical fiber is one of modern topics of great importance in the field of optical communication. The solution of numerical analysis of optical solitons in Birefringent Fibers with Kerr law nonlinearity for special cases Thirring solitons. The ADM was applied to Thirring solitons and revealed that the method gives accurate solutions. (Bakodah et al. 2017).

ADM was a special case of Lyapunovs artificial small parameter method. ADM and Lyapunovs artificial small parameter method were two popular analytic methods for solving nonlinear differential equations. ADM also can derived from the more general Lyapunovs artificial small parameter method. Lyapunovs artificial small parameter method was a powerful analytic method for solving nonlinear problem. (Zhang & Liang 2015).

Danish and Mubashshir (2016), the authors have proposed an ADM based on algorithm and properties of Shank's transformation that not only exploits the nonlinearity handling characteristics of ADM but also it intelligently utilizes the quick convergence.

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