SYNCHRONIZATION FOR DIFFERENT OPINIONS IN MALAYSIA MULTIRACIAL SOCIETY: A MATHEMATICAL EXPLORATION STUDY

HASNA HENG BINTI KAMAL KOH

A thesis submitted in

fulfillment of the requirement for the award of the degree of

Master of Science

Faculty of Science, Technology and Human Development

Universiti Tun Hussein Onn Malaysia

MARCH 2017

ACKNOWLEDGEMENT

All praises to Allah the Almighty, the Most Beneficent and Most Merciful, for countless bounty, wisdom and guidance. Peace be upon The Last Messenger Muhammad, as the most influential man who brought the light to humans life through vast knowledge and splendid character.

I would like to express my gratitude to my supervisor, Dr. Phang Chang for giving me the opportunity to complete my Master's degree with this thesis. He has been providing his maximum time and effort to guide and facilitate me in carrying out this study. Apart from that, he also has been very kind and concerned for my well-being in Batu Pahat. My appreciation also goes to my co-supervisor, Dr. Siti Suhana for her assistance when I counter some problems in my research.

A special thanks to the deans, lecturers and staff in Faculty of Science, Technology and Human Development and Centre for Graduate Studies, or more generally in Universiti Tun Hussein Onn Malaysia, for being kind and accommodating to students' needs time to time.

Lastly, I would like to thank my parents and family members, friends and everyone who directly or indirectly involved in the completion of this thesis. May all of them be rewarded honourably.

ABSTRACT

Malaysia is known for the multiracial and multicultural society who lives together harmoniously despite of the diversities. The tolerance of the people towards each others' culture and religions has always been the subject of interest for social science researchers. The harmony is due to the phenomena of synchronization of the notions among the community. Synchronization has been widely studied since it is a natural phenomena happening around us everyday, such as the synchronization of fireflies flashes at night. In relate of these, we anticipate in the study of the synchronization of the opinions in a diverse society with mathematical modelling. In this thesis, we analyse three different methods of synchronization by 3 different models: Kuramoto model, Opinion Changing Rate model and a linear model associate with the famous Friedkin and Johnsen Model. We first develop a modified version of existing mathematical models and then conduct the numerical experiment on the models to utilize them in the desired framework.

ABSTRAK

Malaysia terkenal dengan masyarakat pelbagai kaum dan pelbagai budaya di mana mereka hidup secara harmoni walaupun terdapat pembezaan. Tolenrasi pelbagai kaum ini terhadap budaya dan agama kaum lain sering kali menjadi subjek yang menarik perhatian pengkaji sains sosial. Keharmonian ini adalah disebabkan oleh penyegerakan pegangan dalam kalangan komuniti. Penyegerakan merupakan bidang yang luas dikaji kerana ia merupakan fenomena semulajadi dimana ia berlaku setiap hari di sekitar kita, seperti penyegerakan kerlipan kelip-kelip pada waktu malam. Sehubungan dengan itu, kami menjangka penyegerakan pendapat dalam kalangan masyarakat dengan model matematik. Dalam tesis ini, kami menganalisa tiga kaedah penyegerakan yang berbeza melalui tiga model yang berbeza: model Kuramoto, model kadar pendapat yang berubah-ubah dan model linear yang berkait dengan model Friedkin dan Johnsen. Kami terlebih dahulu membangunkan versi ubahsuai daripada model yang tersedia ada dan kemudian menjalankan eksperimen secara berangka ke atas model tersebut untuk dipraktikkan dalam kerangka kerja yang dihasratkan.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS AND ABBREVIATIONS

α	tuning parameter of the threshold for the exponential factor in OCR
β_i	eigenvalue of modularity matrix B
γ	scale parameter in Lorenztian distribution which specifies the half-width at half-maximum
∂	partial derivative
ϵ	coupling strength between two oscillators
ϵ_c	coupling strength threshold
$\theta_j(t)$	phase angle of j^{th} oscillator at time t
$\dot{\theta}_j(t)$	of j^{th} oscillator at time t
λ	arbitrary constant
π	the numerical value of the ratio of the circumference of a circle to its diameter (approximately 3.14159)
$\rho(\theta, \omega, t)$	probability density
$\rho(\theta, \omega, t)$	probability density
Ω	mean of the unimodal distribution $g(\omega)$
ω_j	natural frequency of oscillator j or natural inclination of change of individual j
$\psi(t)$	average phase of oscillator at time t
A_{jk}	diagonal matrix of an individual's susceptibility
B	modularity matrix

B	vector coefficient of effect of each exogenous variable
C	constant
d	derivative
e	exponential constant
$G(N, E)$	graph of a network with N nodes and E
G, H	cluster
$g(\omega)$	unimodal distribution of the natural frequency or inclination of opinion change
h	step size
I	identity matrix
i	imaginary quantity
j, k, l, m, v	variables
K_{jk}	matrix of interacting strength between individuals
N	population of coupled oscillators or individuals
OCR	opinion changing rate model
ODE	ordinary differential equation
P_{jk}	variable
Q	modularity
$R(t)$	standard deviation of the opinion changing rate
$r(t)$	order parameter at time t
S_{jk}	numerical value for membership variable
s^T	transpose of s
s_j, s_k, g_j, g_k	membership variables that partition a population into two communities
t	time
U	vector of residual scores

$\overrightarrow{u}$	vector of individual's natural inclination of change
u^T	transpose of u
V^t	coefficient of interpersonal effects on opinion change at time t
W_{jk}	matrix of scores on exogenous variable or strength of interaction between individuals
w_{jk}	interpersonal influence between individual j and k
$\dot{X}(t)$	average of opinion changing rates of individual j
$\dot{x_j}(t)$	opinion changing rate of j^{th}individual at time t
$x_j(t)$	opinion of j^{th} individual at time t
Y	vector of outcome scores
z_k	coordinate of oscillator k

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Background

Malaysia is one of the country which the society lives harmoniously in a same region regardless of the diversity in culture and religion. The study on how the diverse populations harmonize to a common thoughts or opinions has always been a major interest in the field of social sciences. Researches in this field are mostly carried out by interviews and questionnaires. It seems that these tools failed to address some issues related to dynamical process, for instance, understanding how the topology of individuals influence the opinion change among the individuals from radically different thoughts, lifestyles, and cultures. Thus, this sociophysics study intended to find a rigorous mathematical model for the dynamical process of the opinion formation process in the context of Malaysia multiracial society by the sense of mathematical analysis.

Particularly, this study intended to investigate the opinion formation and synchronization process of our society. Apart from that, we hope to be able to suggest the condition on which Malaysians will have a better mutual understanding despite the differences among them. With that, perhaps the ideal vision of unity among the Malaysian citizen regardless of their races, religions and cultures can be achieved through this effort. We will achieve this target by firstly develop a modified version of existing mathematical model, the Kuramoto Model and secondly, we study the Opinion Changing Rate model. Thirdly, we discuss some stochastic influence in the process through linear model and Friedkin-Johnsen model. In all the cases, we will be able to conduct the numerical experiment on the model to utilize it in the desired framework.

1.2 Problem Statement

In the field of sociophysics study, there is no suitable mathematical model to describe the opinion changing process in the multiracial society of Malaysia. The research prior to this is mostly conducted through interview or questionnaire which is mere a social study. Hence, this is our aim to have a simple yet rigorous model to describe the opinion changing process in the context of multiracial society in Malaysia.

1.3 Objectives

The main objective of this research is to simulate opinion change by the individuals from radically different opinions in the context of Malaysia multiracial society based on

1. The dynamics of the Kuramoto Model and the modified Kuramoto Model of synchronization.
2. The dynamics of the Opinion Changing Rate Model and the Modified Opinion Changing Rate Model of social network.
3. The linear model adapting the Friedkin-Johnsen Model of influence network.

1.4 Scope of Study

We will focus on three different models, namely Kuramoto model and its modified version, opinion changing rate model and its modified version and modified Friedkin-Johnsen model. Since this is a mathematical based study, we will first derive the formula for Kuramoto model and its modified version. The proving for long run behaviour of Friedkin-Johnsen model is also our focus. However, we omit the derivation of opinion changing rate model. We run the simulation by using Maple 18 and by using at most 100 individuals.

1.5 Main Contribution

Prior researches has offered many efforts in explaining the behaviour of the opinion changing process through analytical experiment. The present work is designed to provide a new idea to the synchronization phenomena, which it focuses more on the dynamics of the synchronization process by the models rather than the exact solution. It is an attempt to have a mathematical model and analysis to describe the opinion changing process in the context of Malaysia society. Thus, the novel effort is the modification made on each model which we adapted, to suit in this context. In addition, we start it fresh with the demonstration of the result graphically through numerical simulations.

1.6 Thesis Outline

This thesis consists of five chapters. First chapter is the brief introduction and background of the study. We also include the objectives drawn for the study and the contributions we might provide upon the study.

Chapter 2 will highlight the necessary information from the previous studies and works done relevant to this research to have a deeper understanding of the scope of this thesis. It summarize the works done previously by the scholars to provide the readers a basic comprehension of the models and the overall idea of the study. It also contains the theory of the methodology to be utilized in carry out the study. Among the topics are introduction of the Kuramoto Model with the mean-field coupling, the proof and its steady solution, introduction of the opinion changing model and Friedkin-Johnsen Model.

In Chapter 3, we will precisely describe the modifications made on the models as the main tool for the experiment. A small study on the linear model also included to have a brief understanding on how the Friedkin-Johnsen model works. We also include some other concept related to the models we intend to use for the research such as modularity and Runge-Kutta 4th order method.

Chapter 4 is about the simulations by the models and their illustrations on graphs. Also, the crucial part of the description and justification on the outcome.

Finally, in Chapter 5 we will summarize the results discussed, conclude the significance of the study and outline some recommendations for the future study in this arena.

CHAPTER 2

LITERATURE REVIEW

2.1 Related Works

The multiracial and multicultural citizen in Malaysia living harmoniously together as neighbours across the country. The tolerance of the people towards each others' culture and religions has always been the subject of interest for researchers. In the early stage, there were studies of multiracial society in various areas such as in higher education (Rabushka, 1969), schools (Santhiram, 1995), economics (Young et al., 1980) and politics (Case, 1993). Recently, there are some focuses on opinion change in different areas, for instance, in political point of view (Mohamad, 2008, Sani et al., 2009) and in industrial sector (Kee, 2005). These studies use the qualitative approach by conducting questionnaires, interviews and observations to obtain specific information.

Interestingly, as early as 1880's, there are researchers in the field of physics and mathematics who started to come out with different approaches to study social behaviour, which is more quantitative and dynamical. This approach of study involving mathematical method in social science is now referred as sociophysics by many. One of the earliest mathematical models developed in relevance to social sciences is the Malthusian model of population growth (Rapoport, 1983). The fascinating argument made by Malthus is that the population growth is a geometric progression while the arable land growth is an arithmetic progression, indicating that the population outstrip the food supply. In 1983, Rapoport wrote a book entitled "Mathematical Models in the Social and Behavioural Sciences" as an effort to demonstrate the integrative function of the mathematical mode of cognition. He aims to restructure the habits of thinking about the social phenomena. In fact, he stated that

"If the ideal of the 'unity of science' bridging both diverse contents

> and cultural differences can be achieved at all, this will be done via mathematization."

In his book, Rapoport grouped the models into three main classes: classical models, stochastic models and structural models. His work has very much contributes to the field of mathematical modelling of social interaction.

In recent years, many sociophysics researchers came out with fruitful approach in the opinion formation study. There are various types of mathematical models and equations being utilize in order to complement with the particular objectives. Among the models are Ising Spin model (Sznajd-Weron and Sznajd, 2000), classical consensus model (Hegselmann and Krause, 2002), game theory (Di Mare and Latora, 2007), time-variant model (Fortunato et al., 2004, Hegselmann and Krause, 2002), Friedkin-Johnsen model (Hegselmann and Krause, 2002), voter model (Krause and Bornholdt, 2012) etc. The most recent is a study done by Ryosuke Yano and Arnaud Martin (Yano and Martin, 2014) in 2014 uses the relativistic Boltzmann-Vlasov type equation in opinion formation study.

2.2 Kuramoto Model

Above and beyond all models mentioned in previous section, Kuramoto model has been the most popular and addressed as most suitable model in studying the synchronization phenomena. This is due to the capacity of the model in solving issues involving large populations (Acebrón et al., 2005, Cooray, 2008, Daniels, 2005, Rogge, 2006, Strogatz, 2000). Kuramoto model has been studied in a wide range of applications, such as chemical reactions (Acebrón et al., 2005), opinion changing (Pluchino et al., 2004), opinion synchronisation (Pluchino et al., 2006), neural synchrony (Lin and Lin, 2009) and so on. A paradigmatic phenomena that always linked to Kuramoto model is the synchronization of fireflies flashing in the forest. In the darkness of night, it can be clearly observed that several fireflies will initially emitting flashes of light incoherently. Amazingly, after a short period of time, the whole swarm of fireflies will emit the flash of light in unison (Acebrón et al., 2005). The uniqueness of this phenomena even become the interest of a community-based public art project to apply the concept by recruiting cyclists and deploys masses of custom bike lights that communicate and synchronize their blinks with one another. They custom made bike lights using LED designed by Chicago-based artist David Rueter (Kim, 2013).

$$\dot{\theta}_j(t) = \omega_j + \frac{\epsilon}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \quad j = 1, \ldots, N,$$

In this model, ϵ is the coupling strength between two oscillators, N is the number of oscillators, $\theta_j(t)$ is the phase (angle) of the j^{th} oscillator at time t, while ω_j is its intrinsic frequency randomly drawn from some unimodal distribution.

The Kuramoto Model is an improved and simplified model of Winfree's work, who formulated the problem of collective synchronization in terms of large population of limit-cycle interaction between oscillators, which is then revised by Japanese physicist Yoshiki Kuramoto in 1975 (Kuramoto, 2003). Kuramoto made analysis on the reduction of the cooperative dynamics of an oscillator community to a phase dynamics. Here, we review the analysis made by Kuramoto plus the refinement made by Strogatz (Strogatz, 2000).

2.2.1 Derivation of Kuramoto Model

Consider a system of N ordinary differential equations undergoing Hopf Bifurcation. The normal form coordinates of each oscillator:

$$\frac{dz_k}{dt} = (\lambda + i\omega_k)z_k - |z_k|^2 z_k + \frac{\epsilon}{N}\sum_{j=1}^{N} z_j, \tag{2.1}$$

where $\lambda > 0$, $\frac{\epsilon}{N}$ is the coupling strength over the population of oscillators. Convert the normal form into polar form by substituting $z_k = r_k e^{i\theta_k}$ to obtain

$$\left(\frac{dr_k}{dt} + r_k i\frac{d\theta_k}{dt}\right)e^{i\theta_k} = (\lambda + i\omega_k)r_k e^{i\theta_k} - |r_k e^{i\theta_k}|^2 r_k e^{i\theta_k} + \frac{\epsilon}{N}\sum_{j=1}^{N} r_j e^{i\theta_j}, \tag{2.2}$$

where $|r_k e^{i\theta_k}|^2 r_k e^{i\theta_k}$ can be reduced to $r_k^3 e^{i\theta_k}$ since

$$|e^{i\theta_k}|^2 = |\cos(\theta_k) + i\sin(\theta_k)|^2 = \left[\sqrt{\cos^2(\theta_k) + \sin^2(\theta_k)}\right]^2 = 1.$$

Replacing $|r_k e^{i\theta_k}|^2 r_k e^{i\theta_k}$ in (2.2) with $r_k^3 e^{i\theta_k}$ to obtain

$$\left(\frac{dr_k}{dt} + r_k i\frac{d\theta_k}{dt}\right)e^{i\theta_k} = (\lambda + i\omega_k)r_k e^{i\theta_k} - r_k^3 e^{i\theta_k} + \frac{\epsilon}{N}\sum_{j=1}^{N} r_j e^{i\theta_j}. \tag{2.3}$$

Then divide above equation with $e^{i\theta_k}$ and we have

$$\frac{dr_k}{dt} + r_k i \frac{d\theta_k}{dt} = \lambda r_k + i\omega_k r_k - r_k^3 + \frac{\epsilon}{N}\sum_{j=1}^{N} r_j e^{i(\theta_j - \theta_k)}. \tag{2.4}$$

Translating the exponential terms into triangular form with Euler's formula depict

$$\frac{dr_k}{dt} + r_k i \frac{d\theta_k}{dt} = \lambda r_k + i\omega_k r_k - r_k^3 + \frac{\epsilon}{N}\sum_{j=1}^{N}(r_j \cos(\theta_j - \theta_k)) + i r_j \sin(\theta_j - \theta_k). \tag{2.5}$$

Separating the real and imaginary parts to obtain the following:

$$\frac{dr_k}{dt} = \lambda r_k - r_k^3 + \frac{\epsilon}{N}\sum_{j=1}^{N} r_j \cos(\theta_j - \theta_k), \tag{2.6}$$

$$\frac{d\theta_k}{dt} = \omega_k + \frac{\epsilon}{N}\sum_{j=1}^{N} \frac{r_j}{r_k} \sin(\theta_j - \theta_k). \tag{2.7}$$

Assume that $\epsilon \ll 1$ and thus $\frac{\epsilon}{N}\sum_{j=1}^{N} r_j \cos(\theta_j - \theta_k) = 0$, (2.6) can be reduced to

$$\frac{dr_k}{dt} = \lambda r_k - r_k^3 \tag{2.8}$$

which is a Bernoulli Equation.

Now we solve (2.8) for all values of k by letting $v = r_k^{-2}$,

$$r_k = v^{-1/2}. \tag{2.9}$$

Differentiate (2.9) with respect to time t to get

$$\frac{dr_k}{dt} = (-\frac{1}{2}) v^{-3/2} \frac{dv}{dt}. \tag{2.10}$$

Hence, substitute (2.9) and (2.10) into Eq.(2.8) to obtain

$$-\frac{1}{2} v^{-3/2} \frac{dv}{dt} - \lambda v^{-1/2} = v^{-3/2}.$$

Hence dividing it by $-v^{-3/2}$ so that

$$\frac{dv}{dt} + 2\lambda v = 2. \tag{2.11}$$

Let $p(t) = 2\lambda$ and applying $\mu(t) = e^{\int p\lambda dt}$, we get

$$\mu(t) = e^{2\lambda t},$$
$$ve^{2\lambda t} = \int 2e^{2\lambda t}dt + c,$$

Assuming $c = 0$, solve the integration to get the solution for r_k as:

$$\begin{aligned} v = \frac{1}{\lambda} = \frac{1}{r_k^2}, \\ r_k \approx \sqrt{\lambda}. \end{aligned} \tag{2.12}$$

The result above indicate that r_k is a constant and thus implies the system of oscillators will rotate in a unit cycle. Let $r_j = r_k$ and $r_j \approx \sqrt{\lambda}$ to get the Kuramoto Model:

$$\frac{d\theta_k}{dt} = \omega_k + \frac{\epsilon}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_k). \tag{2.13}$$

2.2.2 Kuramoto's Analysis

The simulation made by Kuramoto's analysis describes an infinite population of coupled oscillators θ_j given by the dynamics:

$$\dot{\theta}_j(t) = \omega_j + \frac{\epsilon}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \quad j = 1, \ldots, N, \tag{2.14}$$

where ϵ as the coupling strength between two oscillators, N is the number of oscillators, $\theta_j(t)$ is the phase (angle) of the j^{th} oscillator at time t, while ω_j is its intrinsic frequency randomly drawn from some unimodal distribution $g(\omega)$ which is symmetric about a mean Ω, i.e $g(\Omega+\omega) = g(\Omega-\omega)$. In other definition (Rogge, 2006), a distribution $g(\omega)$ symmetric about mean Ω is unimodal if it is nowhere increasing on $[\Omega, \infty)$, i.e.

$$\omega_1 \le \omega_2 \Rightarrow g(\omega_1) \ge g(\omega_2), \quad \forall \omega_1, \omega_2 > \Omega.$$

Generally a Gaussian distribution is widely used by researchers (Daniels, 2005, Strogatz, 2000), while others use a uniform distribution (Pluchino et al., 2006, 2004) and Cauchy or Lorentzian distribution (Acebrón et al., 2005, Jadbabaie et al., 2004). The summation denotes for all the oscillators while the parameter $\epsilon \geqslant 0$ measures the coupling strength. Notice that the factor $\frac{1}{N}$ is to ensure that

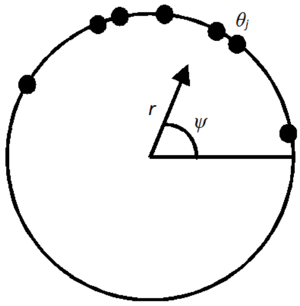

Figure 2.1: Geometric interpretation of the order parameter (2.15).

the model well behaved as $N \to \infty$. The equation depicts a globally coupled system with a mean-field coupling.

Kuramoto defined a complex order parameter to visualize the system through an auxiliary quantity given by:

$$re^{i\psi} = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}. \tag{2.15}$$

The magnitude $0 \le r(t) \le 1$ is a measure of the coherence of the population and $\psi(t)$ is the average phase. It can be interpret geometrically of points rotating around a unit circle in a complex plane (Figure 2.1). We want to rewrite (2.14) in terms of the order parameter. So first we multiply both sides of (2.15) by $e^{-i\theta_j}$ to get

$$re^{i(\psi-\theta_j)} = \frac{1}{N}\sum_{k=1}^{N} e^{i(\theta_k-\theta_j)}$$

By applying the Euler's Formula and equating imaginary parts of above equation we obtain

$$r\sin(\psi-\theta_j) = \frac{1}{N}\sum_{k=1}^{N} \sin(\theta_k-\theta_j).$$

Thus (2.14) yields

$$\dot{\theta}_j(t) = \omega_j + \epsilon r \sin(\psi-\theta_j), \quad i = 1,\dots,N. \tag{2.16}$$

2.2.3 Steady Solutions by Kuramoto Model

Kuramoto first took an effort to guess the solutions in long-term behaviour when the limit $N \to \infty$. Criteria of this steady solution is where r is assumed constant, implies that all the oscillators are effectively independent, and $\psi(t)$ rotates uniformly at frequency Ω. Considering in the rotating frame with Ω at its origin, it is set $\psi \equiv 0$. With the substitution $\omega_j = \omega_j - \Omega$, the governing equation is now

$$\dot{\theta}_j = \omega_j - \epsilon r \sin\theta_j, \quad j = 1, \ldots, N. \tag{2.17}$$

In the limit of infinitely many oscillators, the system is expected to be distributed with a probability density $\rho(\theta, \omega, t)$ with the normalization condition

$$\int_{-\pi}^{\pi} \rho(\theta, \omega, t) d\theta = 1. \tag{2.18}$$

Expressing (2.15) in terms of the probability density, we may get an average over phase and frequency order parameter

$$re^{i\psi} = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} e^{i\theta} \rho(\theta, \omega, t) g(\omega) d\omega d\theta. \tag{2.19}$$

This system will obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta} \{[\omega + \epsilon r \sin(\theta - \psi)] \rho\} = 0. \tag{2.20}$$

Solving this continuity equation with (2.18) and (2.19) to obtain

$$\rho(\theta, \omega) = \frac{C}{|\omega - \epsilon r \sin(\psi - \theta)|} \tag{2.21}$$

where

$$C = \frac{1}{2\pi} \sqrt{\omega^2 - (\epsilon r)^2}. \tag{2.22}$$

By this illustration of order parameter, we can measure the oscillator synchronization when the coupling limit ϵ varies, which resulted in three particular conditions: (i)Not Synchronized, (ii)Partially Synchronized and (iii)Globally Synchronized:

(i) Weak coupling: When $\epsilon \to 0$, Eq. (2.16) gives $\theta_j \approx \omega_j(t) + \theta_j(0)$, $\forall j = 1, \ldots, N$, where subsequently gives $\theta \approx \omega(t)$. Using this to analyse Eq.(2.19), we will find that $r \to 0$ as $t \to \infty$ as we apply the Riemann-Lebesgue lemma. This implies that all oscillators rotate at their natural frequencies and are **not synchronized**.

(ii) Intermediate coupling: When $\epsilon_c < \epsilon < \infty$, accordingly $0 < r < 1$, where part of the oscillators cluster around the mean phase while the others rotating out of synchrony with their own frequency. Thus, the oscillators are **partially synchronized**.

(iii) Strong coupling: When $\epsilon \to \infty$, eventually $\theta_j \approx \psi \ \forall j = 1, \ldots, N$ and Eq. (2.19) denotes $r \to 1$. Thus, all oscillators synchronized to their mean phase, i.e. the oscillators are **globally synchronized**.

Now we use (2.19) and (2.21) to calculate the order parameter in the partial synchronization state:

$$\begin{aligned} r = &\int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} e^{i(\theta-\psi)} \delta(\theta - \psi - \sin^{-1}(\frac{\omega}{\epsilon r})) g(\omega) d\omega d\theta \\ &+ \int_{-\pi}^{\pi} \int_{|\omega|>\epsilon r} e^{i(\theta-\psi)} \frac{C g(\omega)}{|\omega - \epsilon r \sin(\theta - \psi)|} d\omega d\theta \end{aligned} \tag{2.23}$$

Assuming that $g(\omega) = g(-\omega)$ (even function), we have the symmetry $\rho(\theta+\pi, -\omega) = \rho(\theta, \omega)$, which implies the second term will equal to zero. Let us express the above equation in the form below:

$$\begin{aligned} r &= \int_{|\omega|<\epsilon r} \cos(\sin^{-1}(\frac{\omega}{\epsilon r})) g(\omega) d\omega \\ &= \int_{-\pi/2}^{\pi/2} \cos(\theta) g(\epsilon r \sin\theta) \epsilon r \cos\theta d\theta, \\ &= \epsilon r \int_{-\pi/2}^{\pi/2} \cos^2\theta g(\epsilon r \sin\theta) d\theta. \end{aligned} \tag{2.24}$$

Corresponding to the incoherence $\rho = (2\pi)^{-1}$, we always will have a trivial solution $r = 0$. However, for partially synchronize phase the solution satisfying the consistency condition on the amplitude of the order parameter:

$$1 = \epsilon \int_{-\pi/2}^{\pi/2} \cos^2\theta g(\epsilon r \sin\theta) d\theta \tag{2.25}$$

This solution bifurcates continuously from $r = 0$ at $\epsilon = \epsilon_c$ which is the critical coupling:

$$\epsilon_c = \frac{2}{\pi g(0)}. \tag{2.26}$$

In most general study, a Lorenztian density function is utilize for $g(\omega)$:

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)}. \tag{2.27}$$

This allow the explicit calculations of the integrals with exact result after inte-

REFERENCES

Acebrón, J. A., Bonilla, L. L., Vicente, C. J. P., Ritort, F., and Spigler, R. (2005). The kuramoto model: A simple paradigm for synchronization phenomena. *Reviews of modern physics*, 77(1):137.

Case, W. (1993). Semi-democracy in malaysia: withstanding the pressures for regime change. *Pacific Affairs*, pages 183–205.

Cooray, G. (2008). *The Kuramoto Model*. Citeseer.

Daniels, B. C. (2005). Synchronization of globally coupled nonlinear oscillators: the rich behavior of the kuramoto model. *Ohio Wesleyan Physics Dept., Essay*, 7:20.

Di Mare, A. and Latora, V. (2007). Opinion formation models based on game theory. *International Journal of Modern Physics C*, 18(09):1377–1395.

Erbring, L. and Young, A. A. (1979). Individuals and social structure contextual effects as endogenous feedback. *Sociological Methods & Research*, 7(4):396–430.

Fortunato, S., Latora, V., Pluchino, A., and Rapisarda, A. (2004). Vector opinion dynamics in a bounded confidence consensus model. *International Journal of Modern Physics C*, 16(10):1535–1551.

Friedkin, N. E. and Johnsen, E. C. (1990). Social influence and opinions. *Journal of Mathematical Sociology*, 15(3-4):193–206.

Friedkin, N. E. and Johnsen, E. C. (1997). Social positions in influence networks. *Social Networks*, 19(3):209–222.

Friedkin, N. E. and Johnsen, E. C. (1999). Social influence networks and opinion change. *Advances in group processes*, 16(1):1–29.

Gao, J., Sun, Y., Xu, C., and Zheng, Z. (2015). Order parameter analysis for low-dimensional behaviors of coupled phase-oscillators. *arXiv preprint arXiv:1508.00080*.

Hegselmann, R. and Krause, U. (2002). Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3).

Jadbabaie, A., Motee, N., and Barahona, M. (2004). On the stability of the kuramoto model of coupled nonlinear oscillators. In *American Control Conference, 2004. Proceedings of the 2004*, volume 5, pages 4296–4301. IEEE.

Kee, S. L. (2005). *Leadership Styles And Effectiveness Of Change: The Impact Of Resistance To Change*. PhD thesis, USM.

Kim, A. (2013). The kuramoto model(1000 fireflies).

Krause, S. M. and Bornholdt, S. (2012). Opinion formation model for markets with a social temperature and fear. *Physical Review E*, 86(5):056106.

Kuramoto, Y. (2003). *Chemical Oscillations, Waves, and Turbulence*. Dover Books on Chemistry Series. Dover Publications.

Lin, C. and Lin, M. M. (2009). The mathematical research for the kuramoto model of the describing neuronal synchrony in the brain. *Communications in Nonlinear Science and Numerical Simulation*, 14(8):3258–3260.

Mohamad, M. (2008). Malaysia democracy and the end of ethnic politics. *Australian Journal of International Affairs*, 62(4):441–459.

Nagle, R. K., Saff, E. B., and Snider, A. D. (2008). *Fundamentals of Differential Equations*. Pearson Education, Inc, seventh edition.

Newman, M. E. (2006). Finding community structure in networks using the eigenvectors of matrices. *Physical review E*, 74(3):036104.

Ott, E. and Antonsen, T. M. (2008). Low dimensional behavior of large systems of globally coupled oscillators. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 18(3):037113.

Pluchino, A., Boccaletti, S., Latora, V., and Rapisarda, A. (2006). Opinion dynamics and synchronization in a network of scientific collaborations. *Physica*

A: Statistical Mechanics and its Applications, 372(2):316 – 325. Nonlinearity, Nonequilibrium and Complexity: Questions and Perspectives in Statistical Physics A festschrift dedicated to Alberto Robledo on occasion of his 60th birthday.

Pluchino, A., Latora, V., and Rapisarda, A. (2004). Changing opinions in a changing world: A new perspective in sociophysics. *International Journal of Modern Physics C*, 16(04):515–531.

Rabushka, A. (1969). Integration in a multi-racial institution: Ethnic attitudes among chinese and malay students at the university of malaya. *Race & Class*, 11(1):53–63.

Rapoport, A. (1983). *Mathematical models in the social and behavioral sciences.* Wiley New York.

Rodrigues, F. A., Peron, T. K. D., Ji, P., and Kurths, J. (2016). The kuramoto model in complex networks. *Physics Reports*, 610:1–98.

Rogge, J. (2006). *Dynamisch gedrag van oscillatornetwerken Dynamic Behavior of Oscillator Networks.* PhD thesis, Ghent University.

Sani, M. A. M., Yusof, N., Kasim, A., and Omar, R. (2009). Malaysia in transition: A comparative analysis of asian values, islam hadhari and 1malaysia. *Journal of Politics & Law*, 2(3).

Santhiram, R. (1995). Friendship patterns in multi-racial schools: with special reference to a minority community in malaysia. *International Journal of Educational Development*, 15(2):165–173.

Strogatz, S. H. (2000). From kuramoto to crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D: Nonlinear Phenomena*, 143(1):1–20.

Sznajd-Weron, K. and Sznajd, J. (2000). Opinion evolution in closed community. *International Journal of Modern Physics C*, 11(06):1157–1165.

Tay, K. G., Kek, S. L., and Abdul-Kahar, R. (2012). A spreadsheet solution of a system of ordinary differential equations using the fourth-order runge-kutta method. *Spreadsheets in Education (eJSiE)*, 5(2):5.

Yano, R. and Martin, A. (2014). Opinion formation with upper and lower bounds. *arXiv preprint arXiv:1402.7160*.

Young, K., Hasan, P., and Bussink, W. C. (1980). *Malaysia: growth and equity in a multiracial society*. Johns Hopkins University Press.