MHD EFFECTS ON FORCED-CONVECTION STAGNATION NANOFUID
FLOW OVER A STRETCHED SURFACE IN THE PRESENCE OF HEAT
SOURCE/SINK USING SIMILARITY TRANSFORMATION

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A project report submitted in partial fulfillment of the requirement for the award of the degree of Master of Science (Applied Mathematics)

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To

My beloved wife, Laheeb and

My loving daughters, Zainab and Zahraa

And

Especially my loving and supportive mother and father

I have done it for your shining eyes
ACKNOWLEDGEMENT

In the name of Allah, the Beneficent, the Merciful. First of all, I thank ALLAH (SWT), the Lord Almighty, for giving me the health, strength and ability to complete this study and write this thesis.

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ABSTRACT

The magnetohydrodynamics (MHD) effects on forced-convection stagnation flow of Alumina-water (Al₂O₃-water), Copper-water (Cu-water) and Single Walled Carbon Nanotubes-water (SWCNTs-water) nanofluids over a stretched surface in the presence of heat source/sink is studied that includes consideration of thermal radiation. The governing equations (non-linear PDEs) of continuity, momentum, and energy are transformed into first order ordinary differential equations analytically using similarity transformation. The obtained governing equations have been solved numerically by using fourth or fifth-order Runge-Kutta Fehlberg method with shooting technique. The independent software in this thesis is MAPLE 18 programming, which is used to analysis the set of equations and gets results in plot form. The influence on velocity and temperature is discussed for significant parameters such as the magnetic parameter ($M$), radiation parameter ($Rd$), volume friction of nanofluid ($\gamma$), velocity ratio parameter ($\lambda$), and temperature index parameter ($n$) of the flow and heat transfer characteristics. Salient features of the results are analyzed and discussed.
ABSTRAK

Kesan magnetohydrodynamic (MHD) terhadap daya olakan aliran genangan bagi bendalir nano Alumina-air (Al₂O₃), kuprum-air (Cu) dan SWCNTs-air ke atas permukaan yang diregang dengan kehadiran sumber haba telah dikaji yang merangkumi siharan terma. Persamaan menakluk (PDEs tak linear) bagi keselanjuran, momentum dan tenaga ditukarkan kepada persamaan pembezaan biasa peringkat pertama secara analitik menggunakan penjelmaan keserupaan. Persamaan yang diperolehi diselesaikan secara berangka menggunakan kaedah Runge-Kutta Fehlberg keempat kelima dengan teknik meluru. Perisian dalam tesis ini adalah pengaturcaraan MAPLE 18, iaitu digunakan untuk menganalisis set persamaan dan mendapat keputusan dalam bentuk plot. Pengaruh terhadap halaju dan suhu dibincangkan bagi parameter tertentu seperti parameter magnetik, parameter sinaran, isipadu pecahan bendalir-nano, parameter nisbah halaju, dan parameter indek suhu bagi aliran dan ciri-ciri pemindahan haba. Ciri-ciri penting keputusan dianalisa dan dibincangkan.
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Nomenclature

\( A_i, i = 1, \ldots, 5 \) - constants parameters
\( a \) - stretching sheet parameter
\( b \) - free stream velocity parameter
\( C_f \) - skin friction coefficient
\( f \) - dimensionless stream function
\( k \) - thermal conductivity
\( M \) - magnetic parameter
\( Nu \) - Nusselt number
\( Pr \) - Prandtl number
\( Rd \) - Radiation parameter
\( Re_x \) - local Reynolds number
\( T \) - fluid temperature
\( T_\infty \) - ambient temperature
\( (u, v) \) - velocity components in the \((x, y)\) directions, respectively
\( (x, y) \) - Cartesian coordinates along \(x, y\) axes, respectively

Greek symbols

\( \alpha \) - thermal diffusivity
\( \eta \) - similarity parameter
\( \theta \) - similarity function for temperature
\( \rho \) - density
\[ \phi \quad - \quad \text{nanoparticle volume fraction} \\
\mu \quad - \quad \text{dynamic viscosity} \\
\nu \quad - \quad \text{kinematic viscosity} \\
\psi \quad - \quad \text{stream function} \\
\lambda \quad - \quad \text{velocity ratio parameter} \\
\sigma \quad - \quad \text{electrical conductivity} \\
\sigma_e \quad - \quad \text{Stefan-Boltzmann constant} \\
\beta_R \quad - \quad \text{mean absorption coefficient} \\
\]

**Subscripts**

\[ w \quad - \quad \text{condition at the surface} \\
\infty \quad - \quad \text{condition at infinity} \\
nf \quad - \quad \text{nanofluid} \\
f \quad - \quad \text{base fluid} \\
s \quad - \quad \text{nano-solid-particles} \]
CHAPTER 1

INTRODUCTION

1.1 Research Background

Heat transfer across boundary layers of stretching surfaces finds applications in extrusion of plastic sheets, polymers, spinning of fibers, and cooling of elastic sheets. The quality of the final product depends on the rate of heat transfer and therefore cooling procedures must be controlled effectively. The flow of an incompressible viscous fluid over a linearly stretching sheet was first studied by Crane (1970), who obtained an exact solution for the flow field. Afzal & Varshney (1980), proposed a general power-law stretching sheet of the form \( x^m \), where \( x \) is the distance measured from the issuing slit and \( m \) is a constant. The heat transfer from a stretching sheet with variable surface temperature was studied by Afzal (1993), who found exact solutions for specific values of \( m \). Elbashbeshy & Bazid (2004) studied flow and heat transfer in a porous medium over a stretching surface with internal heat generation and suction/blowing when the surface is kept at constant temperature. The study of magnetic field effects has important applications in physics and engineering such as in the cooling of reactors and in many metallurgical processes involving the cooling of contiguous tiles. Also, in several engineering processes, materials manufactured by extrusion processes and heat-treated materials traveling between a feed roll and a wind-up roll on convey belts possess the characteristics of a moving continuous surface. In recent years, we find several applications in the polymer industry (where one deals with stretching of plastic sheets) where hydromagnetic techniques are being used. In view
of these applications, Chakrabarti & Gupta (1979), studied the magnetohydrodynamic (MHD) flow of Newtonian fluids initially at rest, over a stretching sheet at different uniform temperatures. Borkakoti & Bharali (1983), studied the two-dimensional channel flows with heat transfer analysis of a hydromagnetic fluid where the lower plate was a stretching sheet.

The term "nanofluid" refers to a liquid containing a suspension of submicronic solid particles (nanoparticles). The term was coined by Choi (1995). MHD effect on natural convection heat transfer in an inclined L-shape enclosure filled with nanofluid was studied by Sheikholeslami, Ganji, Gorji-Bandpy & Soleimani (2014). They found that enhancement in heat transfer has an inverse relationship with Hartmann number and Rayleigh number. Rashidi, et al. (2013) considered the analysis of the second law of thermodynamics applied to an electrically conducting incompressible nanofluid fluid flowing over a porous rotating disk. They concluded that using magnetic rotating disk drives has important applications in heat transfer enhancement in renewable energy systems. Sheikholeslami, Gorji-Bandpy & Soleimani (2013) used heatline analysis to simulate two-phase nanofluid flow and heat transfer. Their results indicated that the average Nusselt number decreases as buoyancy ratio number increases until it reaches a minimum value and then starts increasing. Sheikholeslami, Hatami & Ganji (2014) analyzed the MHD nanofluid flow and heat transfer between two horizontal plates in a rotating system. Their results indicated that, for both suction and injection, Nusselt number is directly proportional to nanoparticle volume fraction. Nanofluid flow and heat transfer has been investigated by several authors (Hatami & Ganji, 2014a,b; Hayat, Shehzad, & Alsaedi, 2012; Hayat, Shehzad, Qasim, & Obaidat, 2012; Mustafa, Hina, Hayat, & Alsaedi, 2012; Shehzad, Alsaedi, & Hayat, 2012; Sheikholeslami & Ganji, 2013, 2014a,b,c; Sheikholeslami & Gorji-Bandpy, 2014; Sheikholeslami, Gorji-Bandpy, & Ganji, 2012; Sheikholeslami, Gorji-Bandpy, & Ganji, 2013; Sheikholeslami, Gorji-Bandpy, & Domairry, 2013; Sheikholeslami, Gorji-Bandpy, & Ganji, 2014; Sheikholeslami, Gorji-Bandpy, Ganji, Rana, & Soleimani, 2014).

In some device applications such as magnetic sensors, a nanofluid can be used

From our current study, we report on the magnetic field effects in forced-convection flow of a nanofluid over a stretching surface. An analytical study of solutions of these equations presents a challenge for mathematicians. This is due to the fact that, as a rule, these equations cannot be reduced to simple equations with known solutions. Here, we demonstrate this fact by using similarity transformation. At first the governing equations will be reduced to system of ordinary differential equations by using similarity transformation, and there-by find the solutions of these equations numerically using maple software. Finally, we will compare between our results and the results published in literature.

1.2 Problem Statement

The challenge is to identify the following problems:

(i) What is the effective method to reduce partial differential equation (PDE) to ordinary differential equation (ODE) analytically?

(ii) What is the effective solution method to analyze the steady forced convection boundary layer flow?

(iii) How to describe the mathematical model of the best solution of medium included heat transfer effect on nanoparticles in the fluid flow?
1.3  **Research Objectives**

The objectives of this research are:

(i) To reduce the governing problem from partial differential equations (PDEs) to ordinary differential equations (ODEs) using the similarity transformation.

(ii) To find the approximate solution of the converting differential equations numerically using the fourth or fifth-order RungeKutta fehlberg method with shooting technique.

(iii) To compare the present numerical result of the velocity profiles with previous published result.

(iv) To investigate the effects of magnetic strength on velocity and temperature of the nanofluids.

1.4  **Scope of the Study**

The MHD effects on forced-convection stagnation flow of Alumina-water (Al₂O₃-water), Copper-water (Cu-water) and Single-Walled Carbon Nanotubes-water (SWCNTs-water) nanofluids focus on a stretched surface in the presence of heat source/sink. This research used the similarity transformation to reduce PDE to ODE analytically. By using Maple programming we obtained approximate solution.

1.5  **Thesis Outline**

The first chapter establishes the purpose of the research, the statement of the problem, the scope of the research and the method to be used.
Chapter 2 presents a review of the relevant literature dealing with nanoparticles to introduces nanofluid, lorentz force, magneto hydro dynamics, forced convection to introduces mechanism of forced convection.

Chapter 3 presents the method that will be used along the analysis, how to reduce the governing equation to the dimensionless form, some surface introduction of calculation and some introduction of MAPLE 18 command. It also shows the equations that are used in the shooting technique with MAPLE 18. Chapter 4 presents the result and discussion on what finding that have been obtained from the simulation, how the prescribe parameters effects on velocity and temperature profiles. Chapter 5 presents the conclusion on the objectives that has been achieved. Some recommendations for future work are also mentioned in this chapter. Figure 1.1 shows the project research flow chart. Table 1.1 shows the Gantt chart for Project 1 and that gives the outline and durations to complete the task. Master Project 1 cover on Chapter 1 until Chapter 3. Meanwhile, Table 1.2 is the Gantt chart for Master Project 2 and this cover the whole project but more focus on the results that have been obtained.
1.6 Flowchart of research

The summary of the activity along the research is shown by flow chart in Figure 1.1.

START

INTRODUCTION
- Objective and project
- Scope of project

LITERATURE REVIEW
- Journal review
- Gathered information

METHODOLOGY
- Problem Formulation
- Reduce PDE to ODE Using Similarity Transformation

ANALYSIS AND ACQUISITION
- Reduce PDE to ODE Using Similarity Transformation
- Calculation and Solving Equation

RESULTS, DISCUSSION AND CONCLUSION

PRESENTATION AND FINAL REPORT SUBMISSION

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Figure 1.1: Project research flow chart
### 1.7 GANTT CHART

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**TABLE 1.1: GANTT CHART PS 1**
1.8 GANTT CHART

Table 1.2 shows the gantt chart for Master Project 2, PS 2.

<table>
<thead>
<tr>
<th>Project 2 Activities</th>
<th>Weeks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1. Registration of new semester.</td>
<td></td>
</tr>
<tr>
<td>2. Updating previous chapters 1, 2 and 3 with new information.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>3. Chapter 4 (Results &amp; Discussion)</td>
<td></td>
</tr>
<tr>
<td>-Introduction</td>
<td></td>
</tr>
<tr>
<td>-Velocity and temperature profiles on Al₂O₃-water, SWCNTs-water and Cu-water</td>
<td></td>
</tr>
<tr>
<td>-Table sleep and velocity heat transfer on Al₂O₃-water, SWCNTs-water and Cu-water</td>
<td></td>
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<tr>
<td>-Companson of velocity profiles with previous published work</td>
<td></td>
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<td></td>
<td>3</td>
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<td>4. Mid semester break (1 week)</td>
<td></td>
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<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td>5. Chapter 5 (Conclusion)</td>
<td></td>
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<tr>
<td>- Conclusion</td>
<td></td>
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<tr>
<td>- Recommendation</td>
<td></td>
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<td></td>
<td>5</td>
</tr>
<tr>
<td>6. Submission report to supervisor and correction</td>
<td></td>
</tr>
<tr>
<td>7. Submission of report to FSTPI</td>
<td></td>
</tr>
<tr>
<td>8. Presentation of final project 2</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 1.2: GANTT CHART PS 2**
CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, our goal is to review some of the results and techniques available in the literatures. This chapter is divided into nine sections: Section 2.2 deals with the nanoparticles, this section help us to introduce nanofluid. While Section 2.3 discusses about lorentz force. Section 2.4 provides a Magnetohydrodynamics. Then Section 2.5 introduces the forced convection, this section includes mechanism of forced convection. Section 2.6 demonstration the stagnation point. While, Section 2.7 dedicated to the notion of Reynolds number. The Prandtl number and Nusselt number in Sections 2.8, and 2.9 respectively.

2.2 Nanoparticles

Nanoparticles are particles between 1 and 100 nanometers in size. In nanotechnology, a particle is defined as a small object that behaves as a whole unit with respect to its transport and properties. Particles are further classified according to diameter (Guardia, 2007). Ultrafine particles are the same as nanoparticles and between 1 and 100 nanometers in size. Coarse particles cover a range between 2,500 and 10,000 nanometers. Fine particles are sized between 100 and 2,500 nanometers.

Nanoparticle research is currently an area of intense scientific interest due to a wide variety of potential applications in biomedical, optical and electronic fields.

Nanoparticles are of great scientific interest as they are effectively a bridge between bulk materials and atomic or molecular structures. A bulk material should
have constant physical properties regardless of its size, but at the nano-scale this is often not the case. The properties of materials change as their size approaches the nanoscale and as the percentage of atoms at the surface of a material becomes significant.

For bulk materials larger than one micrometre the percentage of atoms at the surface is minuscule relative to the total number of atoms of the material.

The interesting and sometimes unexpected properties of nanoparticles are not partly due to the aspects of the surface of the material dominating the properties in lieu of the bulk properties. (Taylor et al., 2013a; Taylor, 2012; Hewakuruppu, 2013; Taylor et al., 2013b).

Figure 2.1: Images of prepared mesoporous silica nanoparticles with mean outer diameter: (a) 20nm, (b) 45nm, and (c) 80nm. (d) image corresponding to (b). The insets are a high magnification of mesoporous silica particle (Guardia, 2007).

2.2.1 Nanofluid

A Nanofluid is a fluid containing nanometer-sized particles, called nanoparticles. These fluids are engineered colloidal suspensions of nanoparticles in a base fluid. The nanoparticles used in nanofluids are typically made of metals, oxides, carbides, or carbon nanotubes. Common base fluids include water, ethylene glycol and oil (Taylor et al., 2013a; Buongiorno, 2006).
Nanofluids have novel properties that make them potentially useful in many applications in heat transfer (Minkowycz et al., 2013), including microelectronics, fuel cells, pharmaceutical processes, and hybrid-powered engines (Das et al., 2007), engine cooling/vehicle thermal management, domestic refrigerator, chiller, heat exchanger, grinding, machining and in boiler flue gas temperature reduction. They exhibit enhanced thermal conductivity and the convective heat transfer coefficient compared to the base fluid (Sadik & Pramuanjaroenkij, 2009). Knowledge of the rheological behavior of nanofluids is found to be very critical in deciding their suitability for convective heat transfer applications (Witharana et al., 2011; Chen et al., 2009).

In analysis such as computational fluid dynamics (CFD), nanofluids can be assumed to be single phase fluids. However, almost all of new academic paper use two-phase assumption. Classical theory of single phase fluids can be applied, where physical properties of nanofluid is taken as a function of properties of both constituents and their concentrations (Maiga et al., 2005). An alternative approach simulates nanofluids using a two-component model (Kuznetsov & Nield, 2009).

The spreading of a nanofluid droplet is enhanced by the solid-like ordering structure of nanoparticles assembled near the contact line by diffusion, which gives rise to a structural disjoining pressure in the vicinity of the contact line (Wasan & Nikolov, 2003). However, such enhancement is not observed for small droplets with diameter of nanometer scale, because the wetting time scale is much smaller than the diffusion time scale (Lu et al., 2013).

### 2.3 Lorentz Force

Lorentz force, the force exerted on a charged particle $q$ moving with velocity $v$ through an electric $E$ and magnetic field $B$. The entire electromagnetic force $F$ on the charged particle is called the Lorentz force (after the Dutch physicist Hendrik A. Lorentz) and is given by

$$F = qE + qv \times B.$$  \hspace{1cm} (2.1)

The first term is contributed by the electric field. The second term is the magnetic force and has a direction perpendicular to both the velocity and the magnetic field. The
magnetic force is proportional to $q$ and to the magnitude of the vector cross product $\mathbf{v} \times \mathbf{B}$. In terms of the angle $\theta$ between $\mathbf{v}$ and $\mathbf{B}$, the magnitude of the force equals $qvB \sin \theta$. An interesting result of the Lorentz force is the motion of a charged particle in a uniform magnetic field. If $\mathbf{v}$ is perpendicular to $\mathbf{B}$ (i.e., with the angle $\theta$ between $\mathbf{v}$ and $\mathbf{B}$ of 90), the particle will follow a circular trajectory with a radius of $r = m\mathbf{v}/q\mathbf{B}$. If the angle $\theta$ is less than 90, the particle orbit will be a helix with an axis parallel to the field lines. If $\theta$ is zero, there will be no magnetic force on the particle, which will continue to move undeflected along the field lines. Charged particle accelerators like cyclotrons make use of the fact that particles move in a circular orbit when $\mathbf{v}$ and $\mathbf{B}$ are at right angles. For each revolution, a carefully timed electric field gives the particles additional kinetic energy, which makes them travel in increasingly larger orbits. When the particles have acquired the desired energy, they are extracted and used in a number of different ways, from fundamental studies of the properties of matter to the medical treatment of cancer (Graneau, 2001). The Lorentz force is the combination of electric and magnetic force on a point charge due to electromagnetic fields. If a particle of charge $q$ moves with velocity $\mathbf{v}$ in the presence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, then it will experience a force.

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$$ (2.2)

Variations on this basic formula describe the magnetic force on a current-carrying wire (sometimes called Laplace force), the electromotive force in a wire loop moving through a magnetic field (an aspect of Faraday's law of induction), and the force on a charged particle which might be traveling near the speed of light (relativistic form of the Lorentz force). The first derivation of the Lorentz force is commonly attributed to Oliver Heaviside in 1889 by Nahin (2002), although other historians suggest an earlier origin in an 1865 paper by James Clerk Maxwell (Huray, 2009). Hendrik Lorentz derived it a few years after Heaviside.

2.4 Magnetohydrodynamics

Magnetohydrodynamics (MHD) (magneto fluid dynamics or hydromagnetics) is the study of the magnetic properties of electrically conducting fluids. Examples of
such fluids include plasmas, liquid metals, and salt water or electrolytes. The word magnetohydrodynamics (MHD) is derived from magneto- meaning magnetic field, hydro- meaning liquid, and -dynamics meaning movement. The field of MHD was initiated by Hannes Alfvén (1942), for which he received the Nobel Prize in Physics in 1970. The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. The set of equations that describe MHD are a combination of the Navier-Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism. These differential equations must be solved simultaneously, either analytically or numerically.

The simplest form of MHD, ideal MHD, assumes that the fluid has so little resistivity that it can be treated as a perfect conductor. This is the limit of infinite magnetic Reynolds number. In ideal MHD, Lenz’s law dictates that the fluid is in a sense tied to the magnetic field lines. To explain, in ideal MHD a small rope-like volume of fluid surrounding a field line will continue to lie along a magnetic field line, even as it is twisted and distorted by fluid flows in the system. This is sometimes referred to as the magnetic field lines being "frozen" in the fluid (Priest & Forbes, 2000). The connection between magnetic field lines and fluid in ideal MHD fixes the topology of the magnetic field in the fluid—for example, if a set of magnetic field lines are tied into a knot, then they will remain so as long as the fluid/plasma has negligible resistivity. This difficulty in reconnecting magnetic field lines makes it possible to store energy by moving the fluid or the source of the magnetic field. The energy can then become available if the conditions for ideal MHD break down, allowing magnetic reconnection that releases the stored energy from the magnetic field.

2.5 Forced Convection

Convection is the mechanism of heat transfer through a fluid in the presence of bulk fluid motion. Convection is classified as natural (or free) and forced convection depending on how the fluid motion is initiated. In natural convection, any fluid motion is caused by natural means such as the buoyancy effect, i.e. the rise of warmer fluid
and fall the cooler fluid, whereas in forced convection, the fluid is forced to flow over a surface or in a tube by external means such as a pump or fan (Incropera, 2001).

2.5.1 Mechanism of Forced Convection

Convection heat transfer is complicated since it involves fluid motion as well as heat conduction. The fluid motion enhances heat transfer (the higher the velocity the higher the heat transfer rate).

The rate of convection heat transfer is expressed by Newton’s law of cooling:

\[ Q_{\text{conv}} = h(T_s - T_\infty)(W/m^2) \]

\[ Q_{\text{conv}} = hA(T_s - T_\infty)(W) \]  

(2.3)

The convective heat transfer coefficient \( h \) strongly depends on the fluid properties and roughness of the solid surface, and the type of the fluid flow (laminar or turbulent) (Bahrami, 1965).

![Diagram](image)

Figure 2.2: Forced Convection (Bahrami, 1965)

2.6 Stagnation point

In fluid dynamics, a stagnation point is a point in a flow field where the local velocity of the fluid is zero. Stagnation points exist at the surface of objects in the flow field, where the fluid is brought to rest by the object. The idea of a stagnation point is an idealization. This point is infinitesimally small, and air particles flowing along a streamline which leads into it will slow down on their way. The closer they come
to the stagnation point, the slower they flow, and in the end they never arrive at the 
stagnation point (Clancy, 1975).

Figure 2.3: Schematic of Forced-Convection nanofluid Flow Over a Stretching Surface 
(Sheikholeslami et al., 2015)

### 2.7 Reynolds Number ($Re$)

Osborne Reynolds (1842-1912) investigate the practical use of Reynolds number
concept in 1883 for that it given the name. The Reynolds number is defined as the ratio
of inertial forces to viscous forces and consequently quantifies the relative importance
of these two types of forces for given flow conditions. Reynold numbers arise when
performing dimensional analysis of fluid dynamics. It can be used to determine
dynamics similitude between different experimental cases and characterize different
flow. the flow can be considered as:

(i) Laminar flow occurs at low Reynolds numbers. In this case the viscous forces are
dominant, and are characterized by smooth, constant fluid motion. The values of laminar flow are: $Re < 2000$.

(ii) Transitional Flow occurs at the middle of Reynolds numbers. There are irregular
fluctuations. Intermittent laminar and turbulent flow occur, i.e. phases occur
in the flow in which the flow is laminar and phases in which the flow shows
turbulent characteristics. The values of Transitional flow are: $2000 < Re < 4000$.

(iii) Turbulent flow occurs at high Reynolds numbers. It is dominated by inertial forces, which tend to produce chaotic eddies, vortices and other flow instabilities. The values of Turbulent flow are: $Re > 4000$.

Reynolds number ($Re$) is a dimensionless quantity that is used to help predict similar flow patterns in different fluid flow situations. It can be defined as

$$Re = \frac{\text{Inertial force}}{\text{Viscous forces}} = \frac{\rho v L}{\mu} = \frac{v L}{\nu}$$

where:

- $v$ is velocity of object relative to the fluid.
- $L$ is a characteristic linear dimension.
- $\mu$ is dynamic viscosity of the fluid ($kg/(m.s)$ or $N.s/m^2$)
- $\nu$ is the kinematic viscosity ($\nu = \mu/\rho$)($m^2/s$)
- $\rho$ is the density of the fluid ($kg/m^3$)

(See Figure 2.4).

![Diagram of Reynolds number](image)

Figure 2.4: Reynolds number (Osborne, 1883; Rott, 1990)
Fluid flow is generally chaotic and very small changes to shape and surface roughness which result in very different flows. Nevertheless, Reynolds numbers are a very important guide and are widely used. For more details see (Osborne, 1883; Rott, 1990).

2.8 Prandtl number \((Pr)\)

Ludwig Prandtl produced the concept of the boundary layer. The Prandtl number \((Pr)\) is a dimensionless number, named after the German physicist Ludwig Prandtl, defined as the ratio of momentum diffusivity to thermal diffusivity. He proposed that the viscous effects were negligible everywhere except in a thin layer close to the solid boundary of the body where the no-slip condition had to be fulfilled. The Prandtl number formula is given as:

\[
Pr = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{C_p \mu}{k} = \frac{\nu}{\alpha}
\]  

(2.4)

where:

- \(\nu\): momentum diffusivity (kinematic viscosity),
- \(\alpha\): thermal diffusivity,
- \(\mu\): dynamic viscosity,
- \(k\): thermal conductivity,
- \(C_p\): specific heat,
- \(\rho\): density.

Typical values for \(Pr\) are 7 for water. The first such direct numerical simulations were performed by Rogers et al. in 1986 which present the homogeneous shear flow and by Kim & Moin in 1989 which also study the channel flow of \(Pr = 0.1, 0.71\) and 2.0. The \(Pr\) also has high impacts on the boundary layer interaction of the case study structure.
2.9 Nusselt number \((N_u)\)

Nusselt number close to one, namely convection and conduction of similar magnitude, is characteristic of "slug flow" or laminar flow. It is the ratio of convective to conductive heat transfer across (normal to) the boundary. A larger Nusselt number corresponds to more active convection, with turbulent flow typically in the 100-1000 range. The Nusselt number is a dimensionless number that measures the enhancement of heat transfer from a surface that occurs in a real situation compared to the heat transferred if just conduction occurred:

\[
N_u = \frac{\text{convective heat transfer}}{\text{conductive heat transfer}} = \frac{hL}{k} \tag{2.5}
\]

where \(h\) is the convective heat transfer coefficient of the flow, \(L\) is the characteristic length; \(k\) is the thermal conductivity of the fluid (Mori & Nakayama, 1965).
3.1 Introduction

This chapter explains in detail the working method that been used in this view. Methodology is an important part to determine a guideline, direction and the process of the project working so that it will answer some of the present objective. This chapter include the following: problem formulation, similarity transformation, reduce PDE to ODE using similarity transformation, numerical solution.

3.2 Problem Formulation

Consider the steady, two-dimensional flow of a nanofluid near the stagnation point on a stretching sheet saturated with the application of a surface-normal magnetic field as shown in Figure 2.3. The stretching velocity $U_w(x)$ and the free stream velocity $U_\infty(x)$ are assumed to vary proportionally with the distance $x$ from the stagnation point, i.e., $U_w(x) = ax$ and $U_\infty(x) = bx$, where $a$ and $b$ are constants $a > 0$ and $b \geq 0$. The surface of the sheet is also assumed to be subjected to a prescribed temperature $T_w(x) = T_\infty + cx^n$, where $T_\infty$ is the ambient fluid temperature and $c$ and $n$ are constants with $c > 0$ (heated surface). Further, a uniform magnetic field of strength $B_0$ is assumed to be applied in the positive $y$-direction normal to the sheet. The magnetic Reynolds number is assumed to be small, and thus the induced magnetic field is negligible. Also the effects from thermal radiation are considered in this problem.
3.2.1 Governing Equations

The nanofluid is a two-component mixture assumed to be incompressible, having no chemical reactions and negligible radiative heat transfer, with nano-solid-particles and base fluid in thermal equilibrium with no slippage occurring between them. The thermo physical properties of the Cu-water nanofluid are given in Table 3.1 (Li, 2008).

Table 3.1: Thermo physical properties of water and nanoparticles at room temperature (Li, 2008)

<table>
<thead>
<tr>
<th></th>
<th>( \rho (kg/m^3) )</th>
<th>( C_p (J/kgK) )</th>
<th>( k (W/mK) )</th>
<th>( d_p (nm) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure water</td>
<td>997.1</td>
<td>4179</td>
<td>0.613</td>
<td>-</td>
</tr>
<tr>
<td>Cu-water</td>
<td>6500</td>
<td>540</td>
<td>18</td>
<td>29</td>
</tr>
</tbody>
</table>

Under these assumptions, Sheikholeslami (Sheikholeslami et al., 2015) introduced the governing equations as:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}
\]

\[
\rho_n f \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - U_{\infty} \frac{dU_{\infty}}{dx} \right) = \mu_n f \frac{\partial^2 u}{\partial y^2} + \sigma_n f B_0^2 (U_{\infty} - u), \tag{3.2}
\]

\[
(\rho C_p)_n f \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k_n f \frac{\partial^2 T}{\partial y^2} - \frac{\partial q_r}{\partial y} - Q_0 (T - T_{\infty}), \tag{3.3}
\]

subject to boundary conditions

\[
y = 0: \quad u = U_w(x), \quad v = 0, \quad T = T_w(x) \tag{3.4}
\]

\[
y \to \infty: \quad u \to U_{\infty}(x), \quad T \to T_{\infty}
\]

where \( u \) and \( v \) are the velocity components along the \( x \) and \( y \) axes, respectively and \( T \) the fluid temperature. The radiation heat flux \( q_r \) obeys the Rosseland approximation \( q_r = -(4\sigma_e/3\beta_e)(\partial T^4/\partial y) \), where \( \sigma_e \) and \( \beta_e \) are the Stefan-Boltzmann constant and the coefficient of mean absorption, respectively. Following Raptis (1998), fluid-phase temperature differences are small enough that \( T^4 \) can be expanded in a
Taylor series about temperature $T_c$ with higher-order terms neglected, thereby yielding $T^4 \approx 4T^3_c T - 3T^2_c$.

The effective density $\rho_{nf}$, the heat capacitance $(\rho C_p)_{nf}$, and the electrical conductivity $(\sigma_{nf})$ of the nanofluid are given by Sheikholeslami, Gorji-Bandpy & Ganji, 2012:

\[
\rho_{nf} = \rho_f (1 - \phi) + \rho_s \phi, \tag{3.5}
\]
\[
(\rho C_p)_{nf} = (\rho C_p)_f (1 - \phi) + (\rho C_p)_s \phi, \tag{3.6}
\]
\[
\sigma_{nf} = \frac{3(\sigma_s/\sigma_f - 1)\phi}{(\sigma_s/\sigma_f + 2) - (\sigma_s/\sigma_f - 1)\phi}. \tag{3.7}
\]

Brownian motion has a significant impact on the effective thermal conductivity. Koo & Kleinstreuer (2004) proposed that the effective thermal conductivity is composed of the particles conventional static part and a Brownian motion part. This 2-component thermal conductivity model takes into account the effects of particle size, particle volume fraction, and temperature dependence as well as types of particle and base fluid combinations.

\[
k_{eff} = k_{static} + k_{Brownian}, \tag{3.8}
\]

\[
k_{static} = 1 + \frac{3(k_p/k_f - 1)\phi}{(k_p/k_f + 2) - (k_p/k_f - 1)\phi}, \tag{3.9}
\]

where $k_{static}$ is the static thermal conductivity based on the Maxwell classical correlation. The enhanced thermal conductivity component, generated by micro-scale convective heat transfer of a particles Brownian motion and affected by ambient fluid motion, is obtained via simulating Stokes flow around a sphere (nanoparticle). By introducing two empirical functions ($\beta$ and $f$) (Koo, 2004) combined the interaction between nanoparticles in addition to the temperature effect in the model, leading to:

\[
k_{Brownian} = 5 \times 10^4 \beta \phi \rho_f C_p f \frac{k_b T}{\rho_p \sigma_p} f(T, \phi). \tag{3.10}
\]

In recent years, there has been an increasing trend to emphasize the importance of the interfacial thermal resistance between nanoparticles and based fluids, see for example (Prasher et. al., 2005; Jang & Choi, 2004). The thermal interfacial resistance (Kapitza resistance) is believed to exist in the adjacent layers of the two different
materials; the thin barrier layer plays a key role in weakening the effective thermal conductivity of the nanoparticle.

Li (2008) revisited the model of Koo & Kleinstreuer (2005) and combined $\beta$ and $f$ functions to develop a new $g'$ function which captures the influences of particle diameter, temperature, and volume fraction. The empirical $g'$-function depends on the type of nanofluid. Also, by introducing a thermal interfacial resistance $R = 4^{-8} k m^2/W$, the original $k_p$ in Equation (3.9) was replaced by a new $k_{p,eff}$ in the form:

$$R_f + \frac{d_p}{k_p} = \frac{d_p}{k_{p,eff}}. \quad (3.11)$$

For different based fluids and different nanoparticles, the function should be different. Only water-based nanofluids are considered in the current study. For Cuwater nanofluids, this function takes the form:

$$g'(T, \phi, d_p) = (a_1 + a_2ln(d_p) + a_3ln(\phi) + a_4ln(\phi)ln(d_p) + a_5ln(d_p)^2)ln(T) + (a_6 + a_7ln(d_p) + a_8ln(\phi) + a_9ln(\phi)ln(d_p) + a_{10}ln(d_p)^2), \quad (3.12)$$

with the coefficients $a_i, (i = 010)$ are based on the type of nanoparticles. With these coefficients, Cuwater nanofluids have an $R^2$ of 96 percent and 98 percent, respectively (Li, 2008) (Table 3.2). Finally, the KKL correlation is written as:

$$k_{Brownian} = 5 \times 10^4 \phi \rho_f C_p f \sqrt{\frac{k_\text{eff}}{\rho_f d_p} g'(T, \phi, d_p)}. \quad (3.13)$$

Koo & Kleinstreuer (2005) further investigated laminar nanofluid flow in micro-heat sinks using the effective nanofluid thermal conductivity model they had established. For the effective viscosity in the presence of micro-mixing in suspensions, they proposed:

$$\mu_{eff} = \mu_{static} + \mu_{Brownian} = \mu_{static} + \frac{k_{Brownian} \mu_f}{k_f} \rho f. \quad (3.14)$$

where $\mu_{static} = \mu_f/(1 - \phi)^{2.5}$ is the viscosity of the nanofluid, as given originally by Brinkman (Sheikholeslami, 2015).
Table 3.2: Values of the coefficient for Cu-water nanofluids (Li, 2008)

<table>
<thead>
<tr>
<th>Coefficient values</th>
<th>Cu-water</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>-26.59331084</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-0.403818333</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-33.35168050</td>
</tr>
<tr>
<td>$a_4$</td>
<td>-1.915825591</td>
</tr>
<tr>
<td>$a_5$</td>
<td>6.4218584 × 10^{-2}</td>
</tr>
<tr>
<td>$a_6$</td>
<td>48.40336955</td>
</tr>
<tr>
<td>$a_7$</td>
<td>-9.787756683</td>
</tr>
<tr>
<td>$a_8$</td>
<td>190.245610009</td>
</tr>
<tr>
<td>$a_9$</td>
<td>10.92853865</td>
</tr>
<tr>
<td>$a_{10}$</td>
<td>-0.720099836</td>
</tr>
</tbody>
</table>

3.2.2 Stream Function

Now, by the previous the continuity Eq.(3.1) is satisfied by introducing a stream function $\psi$ such that

$$ u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \quad (3.15) $$

The continuity, momentum and energy equations can be transformed into the corresponding ordinary differential equations by the following transformation:

$$ \eta = \left( \frac{a}{v_f} \right)^{1/2} y, \quad \theta(\eta) = \frac{\psi}{(au_f)^{1/2} x}, \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \quad (3.16) $$
3.3 Similarity Transformation

The system of equations in a steady, compressible, laminar boundary layer is composed of four Fundamental equations. Those are: the continuity equation, the momentum equation, the energy equation, and the equation of state. The solutions of these equations, when solved simultaneously for a 2-dimensional boundary layer, are: the velocity in the $x$ and $y$ direction ($u$ and $v$), the pressure ($p$) and the density ($\rho$). The system of equations is a system of partial differential equations (PDEs) and is usually difficult to solve. Therefore, sophisticated transformation methods, called similarity transformations are introduced to convert the original partial differential equation set to a simplified ordinary differential equation (ODE) set. The solutions of this ordinary differential equation set are usually nondimensionalized velocities and temperature. By principle, these ordinary equations are coupled mathematically and usually can be solved by numerical methods. However, with further appropriate assumptions related to the transport properties (e.g. Prandtl number), and flow conditions (e.g. Mach number, geometry around flow), these ODEs can be uncoupled mathematically or can have simpler forms, almost similar to the forms obtained from the incompressible boundary layer analysis. (e.g. Blasius solution, Falkner-Skan equation). Hence, the simplified ODE set makes it possible to get the solution from the already existing solutions of the incompressible analysis and also reduces the computing time in the numerical analysis. For more details see (Brian, 2012).

3.4 Reduce PDE to ODE Using Similarity Transformation

A characteristic feature of a similarity solution is that two or more independent variables can be combined in a particular way such that the PDE becomes an ODE, or a PDE with a reduced number of independent variables. In this work we study the mathematical foundations for determining a similarity transformation of the independent variables that enables one to transform a PDE into an ODE.

The system of equations in the incompressible boundary layer with forced convection, is a PDE system composed of the continuity, the momentum, and the energy equations. These simultaneous equations can be reduced to two ODEs
REFERENCES


