NUMERICAL METHODS FOR FRACTIONAL DIFFERENTIAL EQUATIONS BY NEW CAPUTO AND HADAMARD TYPES OPERATORS

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To my parents Mr. Toh Beng Hock and Madam Chang Tak Har, UTHM, my friends and my supervisor, Prof. Madya Dr. Phang Chang.

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ABSTRACT

Fractional ordinary differential equation (FODE) and fractional partial differential equation (FPDE) emerges in various modelling of physics phenomena. Over past decades, several fractional derivative and operator has been introduced such as Caputo-Fabrizio operator, Caputo-Hadamard fractional derivative, Marchaud fractional derivative or Caputo fractional derivative. The fractional differential equations defined in these fractional derivatives and operators definition are difficult or impossible to solve analytically. Therefore, we seek after highly accurate numerical scheme in efficient ways such as predictor-corrector method, finite difference scheme and spectral collocation method in this research for FODE and FPDE. Caputo-Fabrizo operator is a definition which is verified that does not fit the usual concept neither for fractional nor for integer derivative integral. The main interest of this operator is having regular kernel and which is a necessity of using a model describing the behavior of classical viscoelastic materials, electromagnetic system and viscoelastic materials. Furthermore, associated integral for Caputo-Fabrizio operator is also presented using Laplace transform and Inverse Laplace transform. Hence, we first introduce predictor-corrector scheme involving Caputo-Fabrizo operator, $\alpha > 0$ which represents higher order of approximation $O(h^r)$, $r = \min(\delta_1 + n,\ \delta_2)$, $n = \lceil \alpha \rceil$, $\delta_1, \delta_2 > 0$ as compared to the order of approximation of the classical Caputo fractional derivative $O(h^r)$, $r = \min(\delta_1 + \alpha,\ \delta_2)$, especially in $\alpha \in (0,1)$. Besides that, the Marchaud fractional derivative is a generalization of the Riemann-Liouville fractional derivative, so, we also first develop predictor-corrector scheme involving the Marchaud fractional derivative for $\alpha \in (0,1)$ with applying its improper integral form of the definition instead of Riemann-Liouville definition. This method is stated order of approximation $O(h^{1+\alpha})$, h is discrete step size. Due to the difficulty of dealing with the logarithmic function kernel in Hadamard fractional derivative, finite difference scheme in Caputo-Hadamard fractional derivative, $0 < \alpha < 1$ via incomplete gamma function is also first derived.

Truncation error is also conducted using Lagrange interpolation, which leads to the order of approximation $O(\Delta x^2)$, Δx is space step size. At the end, spectral collocation method via solving a type of eigenvalue problem involving Laguerre polynomials, $\alpha > 0$ decrease needed computation time by implementing minimum summation step with applying eigenvalue degree Laguerre polynomials in order to obtain the accurate approximate solution. Error analysis and Comparative analysis for illustrative examples of solving these developed method with other existing method are shown.

ABSTRAK

Persamaan pembezaan pecahan biasa (PPPB) dan persamaan pembezaan pecahan separa (PPPS) muncul dalam pelbagai pemodelan fenomena fizik. Selama beberapa dekad yang lalu, beberapa pembezaan dan operator pecahan telah diperkenalkan seperti operator Caputo-Fabrizio, pembezaan pecahan Caputo-Hadamard, pembezaan pecahan Marchaud atau pembezaan pecahan Caputo. Persamaan pembezaan pecahan yang ditakrifkan dalam pembezaan pecahan dan definisi operator ini sukar atau mustahil untuk diselesaikan secara analitik. Oleh itu, kami mencari skema berangka yang sangat tepat dengan kaedah yang cekap seperti kaedah peramal-pembetul, skema perbezaan terhingga dan kaedah kolokasi spektrum dalam penyelidikan ini untuk PPPB dan PPPS. Operator Caputo Fabrizo adalah definisi yang disahkan yang tidak serasi dengan konsep biasa baik untuk pecahan mahupun kamiran terbitan integer. Kepentingan utama operator ini adalah memiliki kernel biasa dan yang merupakan keperluan menggunakan model yang menggambarkan tingkah laku bahan viskoelastik klasik, system elektromagnetik dan bahan viskoelastik. Selain itu, kamiran yang berkaitan untuk operator Caputo-Fabrizio juga diterangkan dengan menggunakan jelmaan Laplace dan jelmaan Laplace songsang. Oleh itu, pertama kalinya, kami memperkenalkan skema peramal-pembetul yang melibatkan operator Caputo-Fabrizo, $\alpha > 0$ mewakili penghampiran peringkat tinggi $O(h^r), \mathrm{r} = \min(\delta_1 + n,\ \delta_2)$, $\mathrm{n} = \lceil\alpha\rceil, \delta_1, \delta_2 > 0$ sepertimana berbanding dengan peringkat penghampiran pembezaan pecahan Caputo klasik $O(h^r), \mathrm{r} = \min(\delta_1 + \alpha,\ \delta_2)$, terutamanya dalam selang $\alpha \in (0,1)$. Selain itu, pembezaan pecahan Marchaud merupakan generalisasi untuk pembezaan pecahan Riemann-Liouville. Dengan itu, kami membangunkan skema peramal-pembetul yang melibatkan pembezaan pecahan Marchaud untuk $\alpha \in (0,1)$ dengan menggantikan definisi Riemann-Liouville dengan menggunakan definisi kamiran improper. Kaedah ini mempunyai penghampiran peringkat $O(h^{1+\alpha})$, h ialah saiz langkah diskrit. Oleh kerana kesukaran menangani kernel fungsi logaritma dalam Pembezaan pecahan Hadamard,

skema perbezaan terhingga dalam pembezaan pecahan Caputo-Hadamard, $0 < \alpha < 1$ melalui fungsi gamma yang tidak lengkap juga dibangunkan untuk pertama kalinya. Ralat pemotongan juga dilakukan dengan menggunakan interpolasi Lagrange, dimana ia membawa kepada peringkat penghampiran $O(\Delta x^2)$, Δx ialah saiz Langkah bagi ruang. Akhir sekali, kaedah kolokasi spektrum dengan menyelesaikan sejenis masalah nilai eigen yang melibatkan Polinomial Laguerre, $\alpha > 0$ dapat mengurangkan masa pengiraan dengan melaksanakan langkah penjumlahan minimum dengan menerapkan nilai eigen berdarjah polynomial Laguerre untuk mendapatkan penyelesaian penghamiran yang tepat. Analisis ralat dan analisis perbandingan untuk contoh penyelesaian bagi kaedah yang dibangunkan oleh kami dengan kaedah lain juga ditunjukkan.

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LIST OF SYMBOLS AND ABBREVIATIONS

J f(x)	-	Integer order integral of function
D^nf(x)	-	Integer order derivative of function
$\Gamma(z)$	-	Gamma function
$\mathbb{Z}^+$	-	Positive integer
$\mathbb{R}_+^*$	-	Nonzero real number
!	-	Factorial function
${}^{RL}D^{\alpha} f(x)$	-	Riemann-Liouville fractional derivative
${}^{RL}D^{-\alpha} f(x)$	-	Riemann-Liouville fractional integral
${}^{H}I^{\alpha} f(x)$	-	Hadamard fractional derivative
${}^{GL}D^{\alpha} f(x)$	-	Grunwald-Letnikov fractional derivaative
${}^{M}D^{\alpha} f(x)$	-	Marchaud fractional derivative
${}^{C}D^{\alpha} f(x)$	-	Caputo-Fabrizo operator
$\mathbb{N}$	-	Natural number
$\mathbb{R}_+$	-	Real number
D^{α_k} f(x)	-	Arbitrary fractional derivative of function
D^{α_k} f(x_k)	-	Arbitrary fractional partial derivative with respect to x_k
D^{α_k} f(x, y, z)	-	The specified arbitrary fractional order derivative of space variables function
$D_k^{\alpha_k}$ f(t)	-	The specified term of arbitrary fractional order derivative of time variable function
$D_k^{\alpha_k}$ f(t, x, y, z)	-	The arbitrary fractional order derivative of specified of space and time variables function

$D_k^{\alpha_k}$ f(t, x, y, z)	-	The arbitrary fractional order derivative of specified of space and time variables function
α_k	-	Arbitrary fractional order derivative for α_k
K(x, s)	-	Integral kernel function of Volterra type
$C^n[0, X]$	-	n −Continuous functions over real space interval [0, X]
$f_\lambda(x)$	-	Eigenfunction of fractional Sturm-Liouville problem
$\mathcal{L}$ f(x)	-	Laplace transform of function f(x)
$\mathcal{L}^{-g}$f(x)	-	Inverse Laplace transform of function f(x)
δ(∗)	-	Unit impulse function
erf(*)	-	Error function
erfc(*)	-	Complementary error function
${}_pF_q$	-	Generalized hypergeometric series
$(a)_k$	-	Pochhammer symbol, a (a + 1) ... (a + k − 1)
$\mathbb{Z}^{*-}$	-	Non-positive integer, 0, -1, -2, …
WhittakerM(κ, μ, z)	-	Whittaker M function
WhittakerW(κ, μ, z)	-	Whittaker W function
$\langle P_n, P_m \rangle$	-	Inner product of two orthogonal polynomials P_n and P_m
$L_{v_i(x)}$	-	Eigenvalue degree Laguerre polynomials
$D^k L_{v_i(x)}$	-	Derivative k of Eigenvalue degree Laguerre polynomials
ϵ_x	-	Absolute error
$\lvert f(x) \rvert$	-	Absolute value of function f(x)
$f^*(x)$	-	Approximation function of f(x)
$\lfloor \alpha \rfloor$	-	Floor (α)
$\lceil \alpha \rceil$	-	Ceil (α)
$\{\alpha\}$	-	Decimal part of α

$[\alpha]$	-	Integer part α
H^1(a,b)	-	Continuous function of first order over interval a to b
$\Gamma(\mathrm{a}, \mathrm{x})$	-	Upper incomplete gamma function
$\gamma(\mathrm{a}, \mathrm{x})$	-	Lower incomplete gamma function
$f_0^{(i)}$	-	Initial condition of i derivative of f(x)
$T_{n-1}(x)$	-	Taylor expansion of f(x)
$K_{n,p}^{Re}$	-	Constant that depend on n and p
K_n^{Tr}	-	Constant that depend on n
$\max\lvert f(x)\rvert$	-	Maximum absolute value of f(x)
$\min(f(x))$	-	Minimum value of f(x)
$O(h^r)$	-	Local truncation error
$\mathrm{u}(\mathrm{x}, \mathrm{c})$	-	Bounded condition for time-space equation
$\mathrm{u}(\mathrm{a}, \mathrm{t})$	-	Boundary condition for time-space equation
h	-	Uniform step size
N	-	Number of points
FODE	-	Fractional ordinary differential equation
FPDE	-	Fractional partial differential equation
FSLP	-	Fractional Sturm-Liouville problem

CHAPTER 1

INTRODUCTION

1.1 Background

Fractional calculus is an extension of integer order calculus which involves integral $Jf(x)$ and n^{th} order derivative of function $D^n f(x)$, $n \in \mathbb{Z}^+$. For the integral of function $Jf(x)$, it can be defined as:

$$J\, f(x) = \int_a^x f(s)ds \tag{1.1}$$

Moreover, n^{th} order derivative of the function with respect to x is given by:

$$D^n f(x) = \frac{d^n}{dx^n} f(x) \tag{1.2}$$

Specifically, the first derivative of function $\frac{d}{dx} f(x)$ measures the rate of change of function with respect to the change of x. In other words, the gradient of a line over the interval is well-known.

$$\frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \tag{1.3}$$

Leibniz and L-Hospital began to initiate the idea of generalizing integer order derivative to fractional order derivative in 1695. Some great mathematicians such as Lagrange, Laplace, Lacroix, Fourier, Riemann, Green, Holmgren, Grunwald, Letnikov, Sonini, Laurent, Nekrassov, Krug, Weyl and others had developed subsequent related fractional derivative, $\alpha \in \mathbb{R}_+^*$ (Oldham & Spanier, 1974; Miller & Ross, 1993; Kilbas *et al.*, 2006).

Lacroix first discovered the fractional order derivative formula from integer order derivative of a power series, $k, n \in \mathbb{Z}^+$ (1.4)

$$D^n x^k = \frac{k!}{(k-n)!} x^{k-n}, k \geq n \tag{1.4}$$

By replacing the factorial parts with Gamma function, $(n-1)! = \Gamma(n)$

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \quad R(z) > 0 \tag{1.5}$$

Equation (1.4) can be easily generalized to the fractional order derivative of x^k as shown, $k, \alpha \in \mathbb{R}_+^*$:

$$D^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha}, \quad k \geq \alpha \tag{1.6}$$

In the case of fractional order, $\alpha = 1/2$ and the power of function, $k = 1$

$$\begin{aligned} D^{\frac{1}{2}}\,(x) &= \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-\frac{1}{2}} \\ &= 2\sqrt{\frac{x}{\pi}} \end{aligned} \tag{1.7}$$

These initial of theoretical developments were discovered by Abel, Liouville, Greer, Fourier, and Riemann (Gorenflo & Vessella, 1991; Greer, 1858; Kilbas *et al.*, 1993). Moreover, Sonin and Letnikov (Sonin, 1869; Letnikov, 1872) applied n^{th} order derivative of Cauchy's integral formula:

$$D^n f(z) = \frac{n!}{2\pi i} \int_C \frac{f(\gamma)}{(\gamma - z)^{n+1}} d\gamma \tag{1.8}$$

Thus, the well-known Riemann-Liouville fractional derivative (Letnikov, 1872) as shown in (1.9) was formed which originated from generalizing expression (1.8) to fractional order case

$$^{RL}D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1}\, f(t)\, dt, \;\; \alpha > 0 \tag{1.9}$$

where $n - 1 < \alpha < n$.

Laurent extended the given idea of closed circuit of Sonin and Letnikov to open circuit which is known as the Laurent loop, and the Riemann-Liouville fractional integral definition is defined as, according to Lazarević *et al.* (2014):

$$^{RL}D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \;\; \alpha > 0 \tag{1.10}$$

Moreover, Grunwald and Letnikov formulated a new fractional derivative given as a limit of arbitrary order difference quotient instead of general first principle of differentiation. Today, it is called the Grunwald-Letnikov fractional derivative in Loverro

(2004).

$$^{GL}D^{\alpha}f(x) = \lim_{h\to 0}\frac{\triangle_h^{\alpha}f(x)}{h^{\alpha}} = \lim_{h\to 0}\frac{\sum_{j=0}(-1)^j\binom{\alpha}{j}f(x-jh)}{h^{\alpha}}, \alpha > 0 \tag{1.11}$$

where the binomial coefficients are denoted by:

$$\binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!}$$

Besides, Marchaud provided an integral version of limit of α order difference quotient (Grunwald-Letnikov derivative) which is known as the Marchaud fractional derivative ((Samko *et al.*, 1993), Chapter 2, Section 5, Page 111).

$$^{M}D^{\alpha}f(x) = \frac{f(x)}{\Gamma(1-\alpha)\,x^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{f(x)-f(x-t)}{t^{1+\alpha}}dt \tag{1.12}$$

In case of the axis $R^1 \in (-\infty, \infty)$, if the function $f(x) \in L_p(R^1)$ is a bounded continuous function, then Marchaud fractional derivative $^{M}D^{\alpha}f(x)$ will work for all arbitrary values of $p \in [1, \infty)$.

$$^{M}D^{\alpha}_{\pm}{}^{M}I^{\alpha}_{\pm}f(x) = f(x) \tag{1.13}$$

where the spaces $L_p(R^1)$ is the set of Lebesgue measurable function $f(x)$, complex valued in general for which $\int_{-\infty}^{\infty}|f(x)|^p\,dx < \infty$. While the Riemann-Liouville fractional derivative $^{RL}D^{\alpha}f(x)$ fit the case $p = 1$ only in the frames of L_p spaces, $f(x) \in L_p(R^1)$.

$$^{RL}D^{\alpha}_{\pm}{}^{RL}I^{\alpha}_{\pm}f(x) = f(x) \tag{1.14}$$

Since

$$^{RL}D^{\alpha}_{+}{}^{RL}I^{\alpha}_{+}f(x) = \frac{d}{dx}\int_{-\infty}^{x} f(t)dt \tag{1.15}$$

which assumes summability of $f(x)$ at infinity. In case of $p > 1$, we shall use Marchaud fractional derivative, instead of Riemann-Liouville fractional derivative, treating them as convergent in the norm of $L_p(R^1)$ which is a bounded continuous function.

Apart from that, Caputo reformulated Riemann-Liouville fractional derivative which refers to convolution integral is taken before $n-$ derivative instead of taking $n-$ derivative first before convolution-type integral. Caputo fractional derivative provided memory effect of initial condition (Caputo, 1967), given by:

$$^{C}D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\,\frac{d^n}{dt^n}f(t)\,dt,\ \alpha > 0,\ n = \lceil\alpha\rceil \tag{1.16}$$

In 2015, Michele Caputo and Mauro Fabrizio (Caputo & Fabrizio, 2015) proposed the new Caputo-Fabrizo operator by replacing $(x-t)^{n-\alpha-1}$ with $e^{\frac{-\{\alpha\}(x-t)}{1-\{\alpha\}}}$ and $\frac{1}{\Gamma(n-\alpha)}$ with $\frac{M(\{\alpha\})}{1-\{\alpha\}}$

based on Caputo fractional derivative definition in equation (1.16), with the assumption of $f^{(s)}(a) = 0,\ s = 1, 2, \cdots, [\alpha]$ for $n-1 < \alpha < n$, $\alpha \in \mathbb{R}_+$. The main difference between Caputo-Fabrizo operator and Caputo fractional derivative definition is that, contrary to the classical Caputo fractional derivative with the singular kernel, the exponential kernel in the Caputo-Fabrizo operator doesn't have singularity for $x = t$. In fact, the main interest of this new operator is due to the necessity of using a model describing the behaviour of classical viscoelastic materials, electromagnetic systems and viscoelastic materials, etc.

$$ {}^{CF}D^{\alpha} f(x) = \frac{M(\{\alpha\})}{1-\{\alpha\}} \int_a^x f^{[\alpha]+1}(t) e^{\frac{-\{\alpha\}(x-t)}{1-\{\alpha\}}} dt \tag{1.17} $$

where $\lfloor \alpha \rfloor$, $\lceil \alpha \rceil$, $\{\alpha\}$, $[\alpha]$ are the floor(α), ceil(α), decimal part and integer part of α respectively, and $M(\alpha)$ is a normalization function such that $M(\{\alpha\}) = 1$.

On the other hand, analogous to the Riemann-Liouville derivative using integration first before differentiating the integral kernel, Hadamard fractional derivative, ${}^{H}_{a}D^{\alpha}_{x} f(x)$ for $\alpha > 0$ and $f(x) \in H^1(a, b)$, $a \geq 1$ is defined as (Kilbas *et al.*, 2006):

$$ {}^{H}D^{\alpha} f(x) = \left(x \frac{d}{dx} \right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{n-\alpha-1} \frac{f(t)}{t} dt \tag{1.18} $$

Due to the difficulties of taking integration as a part in solving numerical approximation of Hadamard fractional derivative (1.18), a natural extension of Caputo fractional derivative and Hadamard fractional derivative is to define the Caputo-Hadamard fractional derivative of order $\alpha > 0$, ${}^{CH}D^{\alpha} f(x)$ where $f(x) \in H^1(a, b)$, $a \geq 1$ by Jarad *et al.* (2012).

$$ {}^{CH}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\ln \left(\frac{x}{t} \right) \right)^{n-\alpha-1} t^{n-1} \left(\frac{d}{dt} \right)^n f(t) dt, \quad n = \lceil \alpha \rceil \tag{1.19} $$

Recently, the study of fractional derivatives/integral in the field of fractional calculus is essential. This is because fractional calculus has many applications in various fields, such as fractional conservation of mass, groundwater flow problem, fractional advection dispersion equation, time-space fractional diffusion equation models, structural damping models, PID controllers, acoustical wave equations for complex media, fractional Schrödinger equation in quantum theory and variable-order fractional Schrödinger equation (Wheatcraft & Meerschaert, 2008; Laskin, 2000; Bhrawy & Zaky, 2017; Ray, 2016; Åström & Hägglund, 1995; Hendricks Franssen & Kinzelbach, 2008; Benson *et al.*, 2000; Jiang *et al.*, 2012; Bhrawy & Zaky, 2015; Zareian & Medina, 2010).

Fractional differential equation is an equation that involves fractional derivatives of an unknown function, which can be roughly categorized as:

1. Fractional ordinary differential equation that involves only fractional derivatives with respect to a single independent variable, e.g.
   i. Time Fractional Ordinary Differential Equation, $D_t^\alpha f(t)$
   ii. Space Fractional Ordinary Differential Equation, $D_x^\alpha f(x)$
2. Fractional partial differential equation that involves fractional derivatives with respect to more than one independent variable, e.g.
   i. Time Fractional Partial Differential Equation, $D_t^\alpha f(t,x,y,z)$
   ii. Space Fractional Partial Differential Equation, $D_x^\alpha f(t,x,y,z)$
   iii. Space-Time Fractional Partial Differential Equation

$$D_t^\alpha f(t,x,y,z), D_x^\alpha f(t,x,y,z), D_y^\alpha f(t,x,y,z), D_z^\alpha f(t,x,y,z), \ldots$$
$$D_{tt}^{\alpha+\beta} f(t,x,y,z), D_{xt}^{\alpha+\beta} f(t,x,y,z), D_{xy}^{\alpha+\beta} f(t,x,y,z), \ldots$$

In this research, our main focuses are on fractional ordinary differential equation (FODE) and fractional partial differential equation (FPDE) related to the Caputo-Fabrizo operator, Caputo-Hadamard fractional derivative, Marchaud fractional derivative and Caputo fractional derivative. There are other types of classical fractional derivative in existing literatures such as: Grünwald–Letnikov derivative, Sonin–Letnikov derivative, Liouville derivative, Hadamard derivative, Riesz derivative, Riesz–Miller derivative, Miller–Ross derivative, Weyl derivative, Erdélyi–Kober derivative and some new fractional derivatives include: Machado derivative, Chen–Machado derivative, Coimbra derivative, Katugampola derivative, Caputo–Katugampola derivative, Hilfer derivative, Hilfer–Katugampola derivative, Davidson derivative, Chen derivative, Atangana–Baleanu derivative and Pichaghchi derivative (Miller & Ross, 1993; Kilbas *et al.*, 1993; Ross, 1975; Gorenflo & Mainardi, 2008; Podlubny, 1998; Kiryakova, 1993; Loverro, 2004; Kilbas *et al.*, 2006; Samko *et al.*, 1993).

Nowadays, many well-known analytical methods and numerical methods have been developed to solve fractional differential equation by many researchers (Li *et al.*, 2011; Guo *et al.*, 2015; Bhrawy & Zaky, 2016). Among that are the linear multistep method, discretization method, spectral method and integral transform method. The linear multistep method (Lubich, 1985) is a combination of the predictor step and corrector step to solve numerically fractional ordinary differential equation in both integer and fractional derivatives order. In the predictor step, function values and its derivative values at previous points are used to get an approximate solution at a subsequent point (predicted value). Then, the obtained predicted value is used in the corrector step using implicit method in order to refine the approximation solution at the same subsequent point. The efficiency of this multistep method is in storing and using the value of previous steps as compared to discarding it to get high accuracy.

Common fractional linear multistep methods include fractional Adams–Bashforth methods, fractional Adams–Moulton methods, and fractional backward differentiation formulas. Fractional Adams–Bashforth methods are explicit methods such as the fractional Euler method. Moreover, fractional Adams–Moulton methods which including fractional Trapezoidal rule and fractional backward differentiation formulas are implicit methods. Predictor-corrector so-called Adams–Bashforth–Moulton method (Baskonus & Bulut, 2015) refers to the Adams–Bashforth method and Adams –Moulton method.

In past decades, a detailed error analysis for a fractional Adams–Moulton method has been developed in sense of the fractional Caputo derivative by Diethelm *et al.* (2004). Meanwhile, Caputo-Fabrizo operator has been proposed as a new definition of fractional Caputo derivative. However, there is no fractional Adams–Bashforth –Moulton method that considers the Caputo-Fabrizo operator. Due to the characteristic of this new definition having regular kernel, therefore, Caputo-Fabrizo operator has been applied in developing the Adams–Bashforth–Moulton method which involves the fractional Euler rule and fractional Trapezoidal rule. This method shows high accuracy in solving problems of FODE. The detailed error analysis of this method is also investigated. Besides that, the Marchaud fractional derivative is a generalization of the fractional Riemann-Liouville definition. It has been given less attention compared to the Caputo derivative, Riemann-Liouville derivative or even Caputo-Fabrizo oparator. However, Marchaud derivative regularizes the Riemann-Liouville definition so that it can undergo linear multistep method. Predictor-corrector scheme for solving nonlinear differential equation involving Marchaud derivative and its application will be discussed in this research.

Another method related to the linear multistep method is the discretization method (Liu *et al.*, 2018). Based on the existing literature review, the discretization method uses difference formula as an approximation means in solving linear/ nonlinear fractional ordinary differential equations and fractional partial differential equation. The discretization method is thus a finite difference method which refers to a numerical method by approximating it as a difference equation in discretization form. Difference formulas such as $n-th$ order forward, central and backward difference formula, $n = 1, 2, \ldots$ are applied in solving various FODE and FPDE with Caputo fractional derivative, Caputo-Fabrizo operator, Riemann-Liouville fractional derivative and Hadamard fractional derivative. In 2019, the finite difference scheme related to Caputo-Hadamard fractional derivative had not been introduced by anyone. Therefore, this led us to construct the finite difference scheme for Caputo-Hadamard fractional derivative via incomplete gamma function in solving linear/ nonlinear fractional differential equations.

In addition, the existing literature considers the spectral collocation method of solving multi-term fractional differential equation based on the generalized Laguerre polynomial only in Ghoreishi & Mokhtary (2014). Hence, in the area of fractional ordinary differential equation (FODE), the eigenvalue problem involving Legendre polynomial has been proposed. However, there are no considerations of the spectral collocation method based on solving a type of eigenvalue problem of Laguerre polynomial. Therefore, this prompted us to consider this finding of FODE.

1.2 Problem statement

The fractional differential equations defined in Caputo-Fabrizio operator, Caputo-Hadamard fractional derivative, Marchaud fractional derivative or Caputo fractional derivative are not easy to solve analytically due to the unavailability of analytical solution sometimes, even if analytic solution is available, but it is complex, time-consuming and costly, therefore, we need to develop a numerical scheme such as predictor-corrector method, finite difference scheme and spectral collocation method to tackle the related problem. Analytic methods give an exact result like an integral or exact expression for the solution to get qualitative answer which shows us exactly what happens with each variable while numerical methods are usually more adaptable in approximating result to get quantitative answer by iteratively generate a sequence of approximations to the solution for mathematical problems. In addition, Caputo-Fabrizio operator is relatively new and has a regular kernel by Caputo & Fabrizio (2015). Beside that, the Caputo-Hadamard fractional derivative has an advantage over the Hadamard fractional derivative in dealing with logarithmic function kernel (Jarad *et al.*, 2012) as well as a forgotten history: Marchaud fractional derivative is a generalization of Riemann-Liouville integrals at improper integral form (Samko *et al.*, 1993). There are still relatively limited works which have been done to obtain the simple, reliable, and accurate solution for the problem defined in these operator, therefore, we first develop predictor-corrector scheme involving Caputo-Fabrizo operator which represents higher order of approximation $O(h^r), r = \min(\delta_1 + n, \delta_2), n = \lceil\alpha\rceil$ compared to the order of approximation of the classical Caputo fractional derivative $O(h^r), r = \min(\delta_1 + \alpha, \delta_2)$, especially in $\alpha \in (0, 1)$. Beside that, the Riemann-Liouville fractional derivative definition is invalid for developing its initial value problem which is a main part in predictor-corrector method, so, we also first develop predictor-corrector scheme involving the Marchaud fractional derivative with applying its improper integral form of the definition instead of Riemann-Liouville definition. Due to the difficulty of dealing with the

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