On limitations of the Bruggeman formalism for inverse homogenization

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Abstract. The Bruggeman formalism provides an estimate $\epsilon_{\text{hcm}}^{\text{Br}}$ of the relative permittivity of a homogenized composite material (HCM), arising from two component materials with relative permittivities $\epsilon_a$ and $\epsilon_b$. It can be inverted to provide an estimate of $\epsilon_a$, from a knowledge of $\epsilon_{\text{hcm}}^{\text{Br}}$ and $\epsilon_b$. Numerical studies show that the inverse Bruggeman estimate $\epsilon_a$ can be physically implausible when (i) $\text{Re}\left\{\epsilon_{\text{hcm}}^{\text{Br}}\right\}/\text{Re}\left\{\epsilon_b\right\} > 0$ and the degree of HCM dissipation is moderate or greater; or (ii) $\text{Re}\left\{\epsilon_{\text{hcm}}^{\text{Br}}\right\}/\text{Re}\left\{\epsilon_b\right\} < 0$ regardless of the degree of HCM dissipation. Furthermore, even when the inverse Bruggeman estimate is not obviously implausible, huge discrepancies can exist between this estimate and the corresponding estimate provided by the inverse Maxwell Garnett formalism.

Keywords: Bruggeman; Maxwell Garnett; inverse homogenization; metamaterials

1 INTRODUCTION

A composite material may be regarded as being effectively homogeneous, provided that wavelengths are much larger than the particle sizes of the component materials that make up the composite material. The constitutive parameters of such a homogenized composite material (HCM) can be estimated from a knowledge of the constitutive parameters of its component materials, along with a knowledge of the distributional statistics and shapes of its component particles \cite{1, 2}. The Bruggeman homogenization formalism has been widely applied for this purpose for the past 70 years \cite{1, 3}; and new areas of application for the Bruggeman formalism continue to emerge, for examples, in recent developments pertaining to complex HCMs \cite{4, 5} and negatively–refracting metamaterials \cite{2, 6}. However, a certain limitation of the Bruggeman homogenization formalism came to light in 2004 \cite{7, 8}. In the context of isotropic dielectric HCMs, arising from two component materials with relative permittivities $\epsilon_a$ and $\epsilon_b$, it transpires that the Bruggeman estimate of the HCM relative permittivity may be physically implausible if $\epsilon_a/\epsilon_b < 0$ in the case of nondissipative HCMs or if $\text{Re}\left\{\epsilon_a\right\}/\text{Re}\left\{\epsilon_b\right\} < 0$ in the case of weakly dissipative HCMs. An example of such a problematic homogenization scenario — of considerable interest to the metamaterial community — arises in the homogenization of silver particles with insulating particles at visible and near infrared wavelengths \cite{9}. Let us emphasize that the manifestation of physically–implausible Bruggeman estimates results from the choice of constitutive parameters for the component materials, independently of the distributional statistics or shapes of the component particles. This limitation — which is also relevant to active \cite{10} and anisotropic \cite{11} HCMs — extends to the Maxwell Garnett homogenization formalism which shares a common provenance with the Bruggeman formalism \cite{12}, as well as the Hashin–Shtrikman, Wiener and Bergman–Milton bounds on the HCM’s relative permittivity \cite{7, 8, 13}. Similar anomalous results, for HCMs arising from two component materials with $\text{Re}\left\{\epsilon_a\right\}/\text{Re}\left\{\epsilon_b\right\} < 0$, have been described as ‘electrostatic resonances’ \cite{14–16}, but this term
is avoided in Refs. [7, 9–11, and [13] (and herein) since the estimates of the HCM’s relative permittivity described in Refs. [7, 9–11, and [13] are not physically plausible.

Restricting our attention to the simplest possible case of an isotropic dielectric HCM arising from two isotropic dielectric component materials, in this communication we investigate the applicability of the Bruggeman formalism to the inverse homogenization scenario wherein the relative permittivity of one of the component materials is estimated from a knowledge the relative permittivities of the other component material and the HCM. Formal expressions have been established for the inverse Bruggeman formalism (and the inverse Maxwell Garnett formalism) in the general setting of bianisotropic HCMs [17], but in certain cases these formal expressions may be ill–posed [18] and the ranges of applicability of these inverse formalisms have not been established. Indeed, more generally, inverse problems are commonly ill–posed when the corresponding forward problems are well–posed [19]. The inverse Bruggeman formalism is fundamentally different to the forward Bruggeman formalism. As further described in Sec. 2, for isotropic dielectric materials, an explicit formula provides the inverse formalism estimate of \( \epsilon_a \) or \( \epsilon_b \) whereas the forward formalism estimate of the relative permittivity of the HCM is provided by selecting a root of a quadratic equation. Therefore, the range of applicability of the inverse Bruggeman formalism cannot be inferred from a knowledge of the range of applicability of the forward Bruggeman formalism.

Our study is partly motivated by very recent implementations of the inverse Bruggeman formalism in estimating nanoscale constitutive and morphological parameters of certain sculptured thin films [20], which is a key step in modelling the electromagnetic response of infiltrated sculptured thin films [21, 22].

2 ANALYSIS AND NUMERICAL STUDIES

We consider the homogenization of two isotropic dielectric component materials with relative permittivities \( \epsilon_a \) and \( \epsilon_b \). The component materials \( a \) and \( b \) are assumed to be distributed randomly as spherical particles with volume fractions \( f_a \) and \( f_b = 1 - f_a \), respectively. The Bruggeman estimate of the relative permittivity of the corresponding HCM, namely \( \epsilon_{hcm}^{Br} \), is provided via [3]

\[
\frac{f_a}{\epsilon_a} - \frac{\epsilon_{hcm}^{Br}}{\epsilon_a + 2\epsilon_{hcm}^{Br}} + \frac{f_b}{\epsilon_b} - \frac{\epsilon_{hcm}^{Br}}{\epsilon_b + 2\epsilon_{hcm}^{Br}} = 0,
\]

(1)

which is nonlinear in \( \epsilon_{hcm}^{Br} \). A straightforward manipulation of (1) delivers the explicit formula

\[
\epsilon_a = \frac{(f_a - 2f_b) \epsilon_b + 2\epsilon_{hcm}^{Br} f_a}{f_b (\epsilon_b - \epsilon_{hcm}^{Br}) + f_a (\epsilon_a + 2\epsilon_{hcm}^{Br})} \epsilon_{hcm}^{Br}
\]

(2)

for \( \epsilon_a \) in terms of \( \epsilon_b, \epsilon_{hcm}^{Br}, f_a \) and \( f_b \). Since the component materials \( a \) and \( b \) are treated in an identical manner within the Bruggeman formalism, the corresponding formula for \( \epsilon_b \) has the same form as (2). Notice that as the inverse Bruggeman equation (2) does not involve a square root, there is no scope for \( \text{Im} \{ \epsilon_{hcm}^{Br} \} \) being nonzero if \( \epsilon_b, \epsilon_{hcm}^{Br} \in \mathbb{R} \). This contrasts with the forward Bruggeman formalism where a square root term enables \( \text{Im} \{ \epsilon_{hcm}^{Br} \} \) to be nonzero even though \( \epsilon_a, \epsilon_b \in \mathbb{R} \). This physically–implausible scenario can arise when \( \epsilon_a/\epsilon_b < 0 \) [7, 8].

For comparison, we introduce the Maxwell Garnett estimate of the HCM relative permittivity [3]

\[
\epsilon_{hcm}^{MG} = \epsilon_b + \frac{3f_a \epsilon_b (\epsilon_a - \epsilon_b)}{\epsilon_a + 2\epsilon_b - f_a (\epsilon_a - \epsilon_b)}
\]

(3)

and its corresponding inverse

\[
\epsilon_a = \frac{(2 + f_a) \epsilon_{hcm}^{MG} - 2f_b \epsilon_b}{(1 + 2f_a) \epsilon_b - f_b \epsilon_{hcm}^{MG}} \epsilon_b.
\]

(4)
The limiting behaviour of the inverse Bruggeman estimate (2) as compared with that of the inverse Maxwell Garnett estimate (4) is especially revealing. In the limit \( f_a \to 1 \), both estimates yield the relative permittivity of the HCM, as they must.* In the limit \( f_a \to 0 \), the inverse Bruggeman formalism yields \( \epsilon_a \to -2\epsilon_b \) whereas the inverse Maxwell Garnett formalism yields \( \epsilon_a \to -2\epsilon_b \). While these limits are consistent since \( \epsilon_{hcm}^B = \epsilon_b \) for \( f_a = 0 \), we note that the two inverse estimates of \( \epsilon_a \) may differ markedly for small (but nonzero) values of \( f_a \), provided that \( \epsilon_b \) and the relative permittivity of the HCM are sufficiently different.

We now explore the inverse Bruggeman estimate (2), in comparison with the inverse Maxwell Garnett estimate (4), by means of some illustrative numerical examples. For nondissipative scenarios, the forward Bruggeman formalism runs into difficulties when \( \epsilon_b \) is relatively small, moderate and large. Graphs of the real and imaginary parts of \( \epsilon_a \) yields plausible estimates of the HCM relative permittivity \([7, 8]\). Accordingly, we consider the inverse Maxwell Garnett formalism, versus \( f_a \) are provided for the cases where \( \epsilon_b = \pm 2 \) and \( \epsilon_{hcm}^B = 3 \). When \( \epsilon_{hcm}^B/\epsilon_b > 0 \) the inverse Bruggeman and inverse Maxwell Garnett estimates are in fairly close agreement. However, the values of \( \epsilon_a \) yielded by the two inverse formalisms differ markedly when \( \epsilon_{hcm}^B/\epsilon_b < 0 \), except in the limit as \( f_a \) approaches unity. Most notably, the inverse Bruggeman estimate becomes singular and undergoes a change in sign as the volume fraction increases through \( f_a = 0.56 \), whereas the inverse Maxwell Garnett value remains finite and does not change sign.

![Fig. 1. Plots of \( \epsilon_a \) as determined by the inverse Bruggeman formalism (red, solid curves) and the inverse Maxwell Garnett formalism (blue, dashed curves) versus \( f_a \) for \( \epsilon_b = \pm 2 \) and \( \epsilon_{hcm}^B = 3 \). Estimates of \( \epsilon_a \) delivered by the inverse Maxwell Garnett formalism are strictly valid only for \( f_a \lesssim 0.3 \).](image)

Next we turn to dissipative homogenization scenarios. In the case of the forward Bruggeman formalism, problems arise when \( \text{Re} \{ \epsilon_a \} / \text{Re} \{ \epsilon_b \} < 0 \) and the degree of dissipation is relatively small; if \( \text{Re} \{ \epsilon_a \} / \text{Re} \{ \epsilon_b \} < 0 \) and the degree of dissipation is relatively large or if \( \text{Re} \{ \epsilon_a \} / \text{Re} \{ \epsilon_b \} > 0 \) then the forward Bruggeman formalism was found to deliver physically-plausible estimates of the HCM relative permittivity \([7, 8]\). Accordingly, we consider the regimes where \( \text{Re} \{ \epsilon_{hcm}^B \} / \text{Re} \{ \epsilon_b \} > 0 \) with the degree of dissipation in the HCM being relatively small, moderate and large. Graphs of the real and imaginary parts of \( \epsilon_a \), as estimated by the inverse Bruggeman and inverse Maxwell Garnett formalisms, are plotted versus \( f_a \) in Fig. 2 for the cases \( \epsilon_b = 2 \) and \( \epsilon_{hcm}^B = 3 + \delta i \) where \( \delta \in \{0, 1, 10\} \). When the degree of HCM dissipation is relatively small (\( \delta = 0.1 \)), the estimates of the real and imagi-

*The Maxwell Garnett estimate of the HCM relative permittivity is only strictly applicable in the dilute composite regime \( f_a \lesssim 0.3 \). Accordingly, estimates of \( \epsilon_a \) delivered by the inverse Maxwell Garnett formalism are strictly valid only for \( f_a \lesssim 0.3 \). However, \( \epsilon_{hcm}^M \) coincides with one of the Hashin–Shtrikman bounds on the HCM relative permittivity which applies at all values of \( f_a \) \([23]\).
nary parts of $\epsilon_a$ provided by the inverse Bruggeman and inverse Maxwell Garnett formalisms agree fairly closely. When the degree of HCM dissipation is moderate ($\delta = 1$), there is still fairly close agreement between the inverse Bruggeman and inverse Maxwell Garnett values of $\epsilon_a$ for most values of $f_a$. Crucially, however, for $f_a < 0.05$ the imaginary part of $\epsilon_a$ estimated by the inverse Bruggeman formalism is negative–valued (unlike $\text{Im} \{\epsilon_a\}$ estimated by the inverse Maxwell Garnett formalism which is positive–valued). Here $\text{Im} \{\epsilon_a\} < 0$ is not a physically–plausible outcome as it implies that the homogenization of an active material $a$ and a nondissipative material $b$ results in a dissipative HCM. For both the real and imaginary parts of $\epsilon_a$, the discrepancies between the values estimated by the two inverse formalisms become enormous when the degree of HCM dissipation is relatively large ($\delta = 10$). Furthermore, the inverse Bruggeman estimate is physically implausible for a much larger range of $f_a$ values; i.e., $\text{Im} \{\epsilon_a\}$ estimated by inverse Bruggeman formalism is negative–valued for $f_a < 0.3$ when $\delta = 10$.

Fig. 2. Plots of the real and imaginary parts of $\epsilon_a$ as determined by the inverse Bruggeman formalism (red, solid curves) and the inverse Maxwell Garnett formalism (blue, dashed curves) versus $f_a$ for $\epsilon_b = 2$ and $\epsilon_{hcm}^{Br, MG} = 3 + \delta i$ where $\delta \in \{0.1, 1, 10\}$. Estimates of $\epsilon_a$ delivered by the inverse Maxwell Garnett formalism are strictly valid only for $f_a \lesssim 0.3$. 
Lastly, we explore the $\text{Re} \left\{ \frac{\epsilon_{\text{Br, MG}}}{\epsilon_{\text{hcm}}} \right\} / \text{Re} \{ \epsilon_b \} < 0$ regime. Plots of the real and imaginary values of $\epsilon_a$ in Fig. 3 correspond to the same parameter values as those used for Fig. 2 except that here $\epsilon_b = -2$. The estimates of the inverse Bruggeman formalism are now physically implausible — due to $\text{Im} \{ \epsilon_a \} < 0$ — for a wide range of $f_a$ values, regardless of whether the degree of HCM dissipation is relatively small, moderate or large. In contrast, the estimate of $\text{Im} \{ \epsilon_a \}$ provided by the inverse Maxwell Garnett formalism is positive–valued for all scenarios considered. Additionally, the real parts of $\epsilon_a$ delivered by the two inverse formalisms differ enormously except when $f_a$ approaches unity, for all degrees of HCM dissipation considered.

3 CLOSING REMARKS

In the case of dissipative HCMs, the inverse Bruggeman estimates of $\epsilon_a$ can be physically implausible when

(i) $\text{Re} \left\{ \frac{\epsilon_{\text{Br, MG}}}{\epsilon_{\text{hcm}}} \right\} / \text{Re} \{ \epsilon_b \} > 0$ and the degree of HCM dissipation is moderate or greater; or 

(ii) $\text{Re} \left\{ \frac{\epsilon_{\text{Br, MG}}}{\epsilon_{\text{hcm}}} \right\} / \text{Re} \{ \epsilon_b \} < 0$ regardless of the degree of HCM dissipation.

Fig. 3. As Fig. 2 except that $\epsilon_b = -2$. 


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In the case of nondissipative HCMs, enormous discrepancies can exist between the estimates of \( \epsilon_a \) provided by the inverse Bruggeman formalism and the inverse Maxwell Garnett formalism when \( \epsilon_{\text{hcm}} / \epsilon_b < 0 \). The constitutive parameters chosen in Sec. 2 to illustrate the limitations of the inverse Bruggeman formalism were representative examples. Further numerical studies for other choices of constitutive parameters conforming to scenarios (i) and (ii) (not presented here) yielded qualitatively similar results. Therefore, we conclude that the inverse Bruggeman formalism should be applied with great caution.

Finally, we note that in the very recent implementations of the inverse Bruggeman formalism which motivated this study [20–22], the relative permittivity parameters were positive–valued and the materials were nondissipative. The estimates yielded by the inverse Bruggeman formalism in these cases seem physically plausible, but the acid test can only be provided by suitable experimental measurements.

Acknowledgments

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References

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