Chapter 6

Solving Laminar Boundary Layer Flow along a Stretching Cylinder by Shooting Technique

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Abstract. The steady axisymmetric laminar boundary layer flow along a stretching cylinder immersed in a viscous and incompressible fluid is investigated. The governing partial differential equations are transformed into ordinary differential equations by using similarity transformation. These equations are then solved numerically by a shooting technique in Maple 12 to obtain the velocity and temperature distribution as well as the skin friction coefficient and the local Nusselt number, while the Prandtl number is fixed to unity. Different value of curvature parameters and Prandtl number have been chose in this study to obtain the velocity and temperature profiles.

Keywords. Boundary layer, heat transfer, stretching cylinder, fluid mechanics

1 Introduction

In recent years, considerable efforts have been directed towards the study of the boundary layer flow and heat transfer along a stretching cylinder since it is considered as an important subject in numerous industrial manufacturing processes of fiber technology and extrusion, and theoretical interest. For examples in the cooling of an
infinite metallic plate in a cooling bath, the boundary layer along material handling conveyers, the aerodynamic extrusion of plastic sheets, the boundary layer along a liquid film in condensation processes, paper production, glass blowing, metal spinning and drawing plastic films and polymer extraction. The rate of heat transfer at the stretching surface would affect the quality of the final product. Sakiadis [1] was the first who studied the boundary layer flow on a moving continuous solid surface under constant speed. Crane [2] extended the concept to a stretching sheet with linear velocity and different thermal boundary conditions in Newtonian fluids and present an exact analytical solution.

The heat transfer in laminar boundary layer along a static and moving cylinders has been studied by Lin & Shih[3].Laminar boundary layer flow along a continuously stretching cylinder immersed in a viscous and incompressible fluid has been investigated by Ishak & Nazar [4] and they solved the problem numerically by finite difference method. The present study may be regarded as the extension of the paper by Ishak & Nazar [4]. Thus the results obtained can be compared with these of Ishak & Nazar [4] by using different numerical method.

2 Problem formulation

A steady, axisymmetric boundary layer flow of a viscous and incompressible fluid along a continuously stretching cylinder of diameter 2R as shown in Figure 1 has been considered.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Physical model and coordinate system}
\end{figure}
It is assumed that stretching surface has the velocity \( U(x) \) and the surface temperature \( T_w(x) \) are in the form \( U(x) = U_0(x/L) \) and 
\[ T_w(x) = T_\infty + T_0(x/L)^n, \]
where \( U_0, T_0 \) and \( n \) are constants, which \( n \) refer to the temperature exponent. While \( T_\infty \) and \( L \) are the ambient temperature and characteristics length, respectively.

Under the assumptions along with the boundary layer approximations, the equations of the problem are modelled. The continuity, momentum and energy equations governing such type of flow are written as:

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0, \tag{1}
\]

\[
u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial r} = -\frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \tag{2}
\]

\[
u \frac{\partial T}{\partial x} + \frac{\partial T}{\partial r} = \alpha \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right), \tag{3}
\]

subject to the boundary conditions

\[
u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial r} = -\frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \tag{4}
\]

where \( u \) and \( v \) are the components of velocity in \( x \) and \( r \) directions, respectively, \( T \) is the fluid temperature and \( \alpha \) is the thermal diffusivity.

The continuity equation can be satisfied by introducing a stream function \( \psi(x,r) \) such that \( u = r^{-1} \frac{\partial \psi}{\partial r} \) and \( v = -r^{-1} \frac{\partial \psi}{\partial \xi} \). We introduce the similarity variables as:

\[
\eta = \frac{r^2 - R^2}{2R} \left( \frac{U}{\nu \omega} \right)^{1/2}, \quad \psi = \left( \frac{U \omega x}{\nu \omega} \right)^{1/2} R f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \tag{5}
\]

The momentum and energy equations can be transformed into the corresponding ordinary equations by the following transformation (Mahmood & Merkin, [5]; Ishak & Nazar, [4]). The transformed ordinary differential equations are:

\[
(1 + 2\gamma \eta) f'' + 2\gamma f' + f \omega^2 - f'' = 0, \tag{6}
\]

\[
(1 + 2\gamma \eta) \theta'' + 2\gamma \theta' + Pr (f \theta' - \eta f' \theta') = 0, \tag{7}
\]

The transformed boundary conditions are:
where the primes denotes the differentiation with respect to \( \eta \), and \( \gamma \) is the curvature parameter defined as
\[
\gamma = \left( \frac{\nu L}{U_i R^2} \right)^{1/2}.
\]

The physical quantities of interest are the skin friction coefficient and the local Nusselt number, which are defined as
\[
C_f = \frac{\tau_w}{\rho U^2/2}, \quad \text{Nu}_x = \frac{xq_w}{k(T_w - T_i)},
\]
where the wall shear stress \( \tau_w \) and the wall heat flux \( q_w \) are given by
\[
\tau_w = \mu \left( \frac{\partial u}{\partial r} \right)_{r=R}, \quad q_w = -k \left( \frac{\partial T}{\partial r} \right)_{r=R},
\]
with \( \mu \) and \( k \) being the dynamic viscosity and the thermal conductivity, respectively. Using the similarity variables (5), we obtain
\[
\frac{1}{2} C_f \text{Re}_x^{1/2} = f''(0), \quad \text{Nu}_x / \text{Re}_x^{1/2} = \vartheta'(0),
\]
where \( \text{Re}_x = U_i / \nu \) is the local Reynolds number.

### 3 Results and Discussion

The results were obtained from the shooting method by using Maple 12 software. From this method, we get the values of the skin friction coefficient, \( C_f \), and the local Nusselt number, \( \text{Nu}_x \) for some value of Prandtl number, \( \text{Pr} \), temperature exponent parameter, \( n \) and curvature parameter, \( \gamma \). We perform the results into a table and several graphs. For the validation of the numerical results obtained in this study, the case when the curvature parameter is absent \( (\gamma = 0, \text{flat plate}) \) has been considered and are compared with previously published results as shown in Table 1. The comparison shows a very good agreement.

Figure 1 shows that velocity gradient at the surface is larger for large value of \( \gamma \), which produces larger skin friction coefficient \( f''(0) \). The value of parameters that involve in this problem are Prandtl number, \( \text{Pr} = 1 \) and temperature exponent parameter, \( n = 0 \). We can conclude that the velocity increases as the number of curvature parameter increases.
Figure 2 shows the temperature profiles $\theta(\eta)$ increases when different values of curvature parameter, $\gamma$ are used in increasing order which is 0, 0.1 and 0.2.

**Table 1.** Local Nusselt number $-\theta'(0)$ with various value of $Pr$ and $n$ when the plate is flat, $\gamma = 0$.

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<td>1.3269</td>
<td>4.7679</td>
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</table>

**Table 1.** Local Nusselt number $-\theta'(0)$ with various value of $Pr$ and $n$ when the plate is flat, $\gamma = 0$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Present results $Pr = 1$</th>
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<td>2</td>
<td>1.3333</td>
<td>4.7969</td>
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</tbody>
</table>

**Fig. 1.** Velocity profiles $f'(\eta)$ for some values of $\gamma$  

**Fig. 2.** Temperature profiles $\theta(\eta)$ for some values of $\gamma$

Figure 3 shows similar pattern of temperature profiles $\theta(\eta)$ as in Figure 2 but the temperature exponent parameter, $n$ is fixed to 1. The temperature gradient at the surface increases as the curvature parameter, $\gamma$ increases.
Figure 4 shows the temperature profiles $\theta(\eta)$ for several values of temperature exponent parameter, $n$ with a fixed Prandtl number, $Pr$ and curvature parameter, $\gamma$. The temperature gradient at a surface decreases as the temperature exponent parameter, $n$ increases.

![Fig. 3. Temperature profiles $\theta(\eta)$ for some value of $\gamma$ when $Pr = 1$ and $n = 1$](image1)

![Fig. 4. Temperature profiles $\theta(\eta)$ for some value of $n$ when $Pr = 1$ and $\gamma = 0.2$](image2)

Figure 5 shows the decreasing temperature gradient at a surface as the increasing value of temperature exponent parameter, $n$. It can be observed that higher value of Prandtl number produced smaller boundary layer thickness with the increasing value of temperature exponent temperature.

Figure 6 shows that local Nusselt number increases with Prandtl number, $Pr$, since the higher Prandtl number fluid is a fluid that has a lower thermal conductivity or a higher viscosity which results in thinner thermal boundary layer and hence, higher heat transfer rate at the surface.
Finally, all the Figures show that the boundary condition (8) are satisfied, which support the validity of the numerical results obtained.

4 Conclusions

The present study investigated on how the governing parameters, namely the curvature parameters $\gamma$, Prandtl number $Pr$ and the temperature exponent $n$, influences the boundary layer flow and heat transfer characteristics on the surface of a horizontal cylinder. When $\gamma = 0$ flat plate, the results shows that the local Nusselt number increases as the $Pr$ increases, which agreed with previously reported results. Further, the study on the effects of $\gamma$ on the skin friction coefficient and local Nusselt number reveals that both of them increase as $\gamma$ increases. Thus, the surface shear stress and the heat transfer rate at the surface increase as the curvature parameter increases.
References


