SYNCHRONIZATION FOR DIFFERENT OPINIONS IN MALAYSIA MULTIRACIAL SOCIETY: A MATHEMATICAL EXPLORATION STUDY

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UNIVERSITI TUN HUSSEIN ONN MALAYSIA
SYNCHRONIZATION FOR DIFFERENT OPINIONS IN MALAYSIA MULTIRACIAL SOCIETY: A MATHEMATICAL EXPLORATION STUDY

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ABSTRACT

Malaysia is known for the multiracial and multicultural society who lives together harmoniously despite of the diversities. The tolerance of the people towards each others’ culture and religions has always been the subject of interest for social science researchers. The harmony is due to the phenomena of synchronization of the notions among the community. Synchronization has been widely studied since it is a natural phenomena happening around us everyday, such as the synchronization of fireflies flashes at night. In relate of these, we anticipate in the study of the synchronization of the opinions in a diverse society with mathematical modelling. In this thesis, we analyse three different methods of synchronization by 3 different models: Kuramoto model, Opinion Changing Rate model and a linear model associate with the famous Friedkin and Johnsen Model. We first develop a modified version of existing mathematical models and then conduct the numerical experiment on the models to utilize them in the desired framework.
ABSTRAK

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<tr>
<td>$\alpha$</td>
<td>tuning parameter of the threshold for the exponential factor in OCR</td>
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<tr>
<td>$\beta_i$</td>
<td>eigenvalue of modularity matrix $B$</td>
</tr>
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<td>$\gamma$</td>
<td>scale parameter in Lorenztian distribution which specifies the half-width at half-maximum</td>
</tr>
<tr>
<td>$\partial$</td>
<td>partial derivative</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>coupling strength between two oscillators</td>
</tr>
<tr>
<td>$\epsilon_c$</td>
<td>coupling strength threshold</td>
</tr>
<tr>
<td>$\theta_j(t)$</td>
<td>phase angle of $j^{th}$ oscillator at time $t$</td>
</tr>
<tr>
<td>$\dot{\theta}_j(t)$</td>
<td>of $j^{th}$ oscillator at time $t$</td>
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<tr>
<td>$\lambda$</td>
<td>arbitrary constant</td>
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<tr>
<td>$\pi$</td>
<td>the numerical value of the ratio of the circumference of a circle to its diameter (approximately 3.14159)</td>
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<td>$\rho(\theta, \omega, t)$</td>
<td>probability density</td>
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<td>$\Omega$</td>
<td>mean of the unimodal distribution $g(\omega)$</td>
</tr>
<tr>
<td>$\omega_j$</td>
<td>natural frequency of oscillator $j$ or natural inclination of change of individual $j$</td>
</tr>
<tr>
<td>$\psi(t)$</td>
<td>average phase of oscillator at time $t$</td>
</tr>
<tr>
<td>$A_{jk}$</td>
<td>diagonal matrix of an individual’s susceptibility</td>
</tr>
<tr>
<td>$B$</td>
<td>modularity matrix</td>
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$B$ vector coefficient of effect of each exogenous variable

$C$ constant

$d$ derivative

$e$ exponential constant

$G(N,E)$ graph of a network with $N$ nodes and $E$

$G,H$ cluster

$g(\omega)$ unimodal distribution of the natural frequency or inclination of opinion change

$h$ step size

$I$ identity matrix

$i$ imaginary quantity

$j,k,l,m,v$ variables

$K_{jk}$ matrix of interacting strength between individuals

$N$ population of coupled oscillators or individuals

$OCR$ opinion changing rate model

$ODE$ ordinary differential equation

$P_{jk}$ variable

$Q$ modularity

$R(t)$ standard deviation of the opinion changing rate

$r(t)$ order parameter at time $t$

$S_{jk}$ numerical value for membership variable

$s^T$ transpose of $s$

$s_j,s_k,g_j,g_k$ membership variables that partition a population into two communities

$t$ time

$U$ vector of residual scores
\( \vec{u} \) vector of individual’s natural inclination of change

\( u^T \) transpose of \( u \)

\( V^t \) coefficient of interpersonal effects on opinion change at time \( t \)

\( W_{jk} \) matrix of scores on exogenous variable or strength of interaction between individuals

\( w_{jk} \) interpersonal influence between individual \( j \) and \( k \)

\( \bar{X}(t) \) average of opinion changing rates of individual \( j \)

\( \dot{x}_j(t) \) opinion changing rate of \( j^{th} \) individual at time \( t \)

\( x_j(t) \) opinion of \( j^{th} \) individual at time \( t \)

\( Y \) vector of outcome scores

\( z_k \) coordinate of oscillator \( k \)
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CHAPTER 1

INTRODUCTION

1.1 Background

Malaysia is one of the country which the society lives harmoniously in a same region regardless of the diversity in culture and religion. The study on how the diverse populations harmonize to a common thoughts or opinions has always been a major interest in the field of social sciences. Researches in this field are mostly carried out by interviews and questionnaires. It seems that these tools failed to address some issues related to dynamical process, for instance, understanding how the topology of individuals influence the opinion change among the individuals from radically different thoughts, lifestyles, and cultures. Thus, this sociophysics study intended to find a rigorous mathematical model for the dynamical process of the opinion formation process in the context of Malaysia multiracial society by the sense of mathematical analysis.

Particularly, this study intended to investigate the opinion formation and synchronization process of our society. Apart from that, we hope to be able to suggest the condition on which Malaysians will have a better mutual understanding despite the differences among them. With that, perhaps the ideal vision of unity among the Malaysian citizen regardless of their races, religions and cultures can be achieved through this effort. We will achieve this target by firstly develop a modified version of existing mathematical model, the Kuramoto Model and secondly, we study the Opinion Changing Rate model. Thirdly, we discuss some stochastic influence in the process through linear model and Friedkin-Johnsen model. In all the cases, we will be able to conduct the numerical experiment on the model to utilize it in the desired framework.
1.2 Problem Statement

In the field of sociophysics study, there is no suitable mathematical model to describe the opinion changing process in the multiracial society of Malaysia. The research prior to this is mostly conducted through interview or questionnaire which is mere a social study. Hence, this is our aim to have a simple yet rigorous model to describe the opinion changing process in the context of multiracial society in Malaysia.

1.3 Objectives

The main objective of this research is to simulate opinion change by the individuals from radically different opinions in the context of Malaysia multiracial society based on

1. The dynamics of the Kuramoto Model and the modified Kuramoto Model of synchronization.

2. The dynamics of the Opinion Changing Rate Model and the Modified Opinion Changing Rate Model of social network.

3. The linear model adapting the Friedkin-Johnsen Model of influence network.

1.4 Scope of Study

We will focus on three different models, namely Kuramoto model and its modified version, opinion changing rate model and its modified version and modified Friedkin-Johnsen model. Since this is a mathematical based study, we will first derive the formula for Kuramoto model and its modified version. The proving for long run behaviour of Friedkin-Johnsen model is also our focus. However, we omit the derivation of opinion changing rate model. We run the simulation by using Maple 18 and by using at most 100 individuals.
1.5 Main Contribution

Prior researches has offered many efforts in explaining the behaviour of the opinion changing process through analytical experiment. The present work is designed to provide a new idea to the synchronization phenomena, which it focuses more on the dynamics of the synchronization process by the models rather than the exact solution. It is an attempt to have a mathematical model and analysis to describe the opinion changing process in the context of Malaysia society. Thus, the novel effort is the modification made on each model which we adapted, to suit in this context. In addition, we start it fresh with the demonstration of the result graphically through numerical simulations.

1.6 Thesis Outline

This thesis consists of five chapters. First chapter is the brief introduction and background of the study. We also include the objectives drawn for the study and the contributions we might provide upon the study.

Chapter 2 will highlight the necessary information from the previous studies and works done relevant to this research to have a deeper understanding of the scope of this thesis. It summarize the works done previously by the scholars to provide the readers a basic comprehension of the models and the overall idea of the study. It also contains the theory of the methodology to be utilized in carry out the study. Among the topics are introduction of the Kuramoto Model with the mean-field coupling, the proof and its steady solution, introduction of the opinion changing model and Friedkin-Johnsen Model.

In Chapter 3, we will precisely describe the modifications made on the models as the main tool for the experiment. A small study on the linear model also included to have a brief understanding on how the Friedkin-Johnsen model works. We also include some other concept related to the models we intend to use for the research such as modularity and Runge-Kutta 4th order method.

Chapter 4 is about the simulations by the models and their illustrations on graphs. Also, the crucial part of the description and justification on the outcome.

Finally, in Chapter 5 we will summarize the results discussed, conclude
the significance of the study and outline some recommendations for the future study in this arena.
CHAPTER 2

LITERATURE REVIEW

2.1 Related Works

The multiracial and multicultural citizen in Malaysia living harmoniously together as neighbours across the country. The tolerance of the people towards each others’ culture and religions has always been the subject of interest for researchers. In the early stage, there were studies of multiracial society in various areas such as in higher education (Rabushka, 1969), schools (Santhiram, 1995), economics (Young et al., 1980) and politics (Case, 1993). Recently, there are some focuses on opinion change in different areas, for instance, in political point of view (Mohamad, 2008, Sani et al., 2009) and in industrial sector (Kee, 2005). These studies use the qualitative approach by conducting questionnaires, interviews and observations to obtain specific information.

Interestingly, as early as 1880’s, there are researchers in the field of physics and mathematics who started to come out with different approaches to study social behaviour, which is more quantitative and dynamical. This approach of study involving mathematical method in social science is now referred as sociophysics by many. One of the earliest mathematical models developed in relevance to social sciences is the Malthusian model of population growth (Rapoport, 1983). The fascinating argument made by Malthus is that the population growth is a geometric progression while the arable land growth is an arithmetic progression, indicating that the population outstrip the food supply. In 1983, Rapoport wrote a book entitled “Mathematical Models in the Social and Behavioural Sciences” as an effort to demonstrate the integrative function of the mathematical mode of cognition. He aims to restructure the habits of thinking about the social phenomena. In fact, he stated that

“If the ideal of the ‘unity of science’ bridging both diverse contents
and cultural differences can be achieved at all, this will be done via
mathematization.'

In his book, Rapoport grouped the models into three main classes: clas-
sical models, stochastic models and structural models. His work has very much
contributes to the field of mathematical modelling of social interaction.

In recent years, many sociophysics researchers came out with fruitful ap-
proach in the opinion formation study. There are various types of mathematical
models and equations being utilize in order to complement with the particular
objectives. Among the models are Ising Spin model (Sznajd-Weron and Sznajd,
2000), classical consensus model (Hegselmann and Krause, 2002), game theory
(Di Mare and Latora, 2007), time-variant model (Fortunato et al., 2004, Hegsel-
mann and Krause, 2002), Friedkin-Johnsen model (Hegselmann and Krause,
2002), voter model (Krause and Bornholdt, 2012) etc. The most recent is a
study done by Ryosuke Yano and Arnaud Martin (Yano and Martin, 2014) in
2014 uses the relativistic Boltzmann-Vlasov type equation in opinion formation
study.

2.2 Kuramoto Model

Above and beyond all models mentioned in previous section, Kuramoto model
has been the most popular and addressed as most suitable model in studying the
synchronization phenomena. This is due to the capacity of the model in solving
issues involving large populations (Acebrón et al., 2005, Cooray, 2008, Daniels,
2005, Rogge, 2006, Strogatz, 2000). Kuramoto model has been studied in a wide
range of applications, such as chemical reactions (Acebrón et al., 2005), opinion
changing (Pluchino et al., 2004), opinion synchronisation (Pluchino et al., 2006),
neural synchrony (Lin and Lin, 2009) and so on. A paradigmatic phenomena that
always linked to Kuramoto model is the synchronization of fireflies flashing in the
forest. In the darkness of night, it can be clearly observed that several fireflies will
initially emitting flashes of light incoherently. Amazingly, after a short period of
time, the whole swarm of fireflies will emit the flash of light in unison (Acebrón
et al., 2005). The uniqueness of this phenomena even become the interest of a
community-based public art project to apply the concept by recruiting cyclists
and deploys masses of custom bike lights that communicate and synchronize their
blinks with one another. They custom made bike lights using LED designed by
Chicago-based artist David Rueter (Kim, 2013).
\[ \dot{\theta}_j(t) = \omega_j + \frac{\epsilon}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j), \quad j = 1, \ldots, N, \]

In this model, \( \epsilon \) is the coupling strength between two oscillators, \( N \) is the number of oscillators, \( \theta_j(t) \) is the phase (angle) of the \( j^{th} \) oscillator at time \( t \), while \( \omega_j \) is its intrinsic frequency randomly drawn from some unimodal distribution.

The Kuramoto Model is an improved and simplified model of Winfree’s work, who formulated the problem of collective synchronization in terms of large population of limit-cycle interaction between oscillators, which is then revised by Japanese physicist Yoshiki Kuramoto in 1975 (Kuramoto, 2003). Kuramoto made analysis on the reduction of the cooperative dynamics of an oscillator community to a phase dynamics. Here, we review the analysis made by Kuramoto plus the refinement made by Strogatz (Strogatz, 2000).

### 2.2.1 Derivation of Kuramoto Model

Consider a system of \( N \) ordinary differential equations undergoing Hopf Bifurcation. The normal form coordinates of each oscillator:

\[ \frac{dz_k}{dt} = (\lambda + i \omega_k) z_k - |z_k|^2 z_k + \frac{\epsilon}{N} \sum_{j=1}^{N} z_j, \quad (2.1) \]

where \( \lambda > 0, \frac{\epsilon}{N} \) is the coupling strength over the population of oscillators. Convert the normal form into polar form by substituting \( z_k = r_k e^{i \theta_k} \) to obtain

\[ \left( \frac{dr_k}{dt} + r_k i \frac{d\theta_k}{dt} \right) e^{i \theta_k} = (\lambda + i \omega_k) r_k e^{i \theta_k} - |r_k e^{i \theta_k}|^2 r_k e^{i \theta_k} + \frac{\epsilon}{N} \sum_{j=1}^{N} r_j e^{i \theta_j}, \quad (2.2) \]

where \( |r_k e^{i \theta_k}|^2 r_k e^{i \theta_k} \) can be reduced to \( r_k^3 e^{i \theta_k} \) since

\[ |e^{i \theta_k}|^2 = |\cos(\theta_k) + i \sin(\theta_k)|^2 = \left[ \sqrt{\cos^2(\theta_k) + \sin^2(\theta_k)} \right]^2 = 1. \]

Replacing \( |r_k e^{i \theta_k}|^2 r_k e^{i \theta_k} \) in (2.2) with \( r_k^3 e^{i \theta_k} \) to obtain

\[ \left( \frac{dr_k}{dt} + r_k i \frac{d\theta_k}{dt} \right) e^{i \theta_k} = (\lambda + i \omega_k) r_k e^{i \theta_k} - r_k^3 e^{i \theta_k} + \frac{\epsilon}{N} \sum_{j=1}^{N} r_j e^{i \theta_j}. \quad (2.3) \]
Then divide above equation with \( e^{ik} \) and we have

\[
\frac{dr_k}{dt} + r_k i \frac{d\theta_k}{dt} = \lambda r_k + i\omega r_k - r_k^3 + \epsilon \sum_{j=1}^{N} r_j e^{i(\theta_j - \theta_k)}.
\]  

(2.4)

Translating the exponential terms into triangular form with Euler’s formula depict

\[
\frac{dr_k}{dt} + r_k i \frac{d\theta_k}{dt} = \lambda r_k + i\omega r_k - r_k^3 + \epsilon \sum_{j=1}^{N} (r_j \cos(\theta_j - \theta_k)) + ir_j \sin(\theta_j - \theta_k).
\]  

(2.5)

Separating the real and imaginary parts to obtain the following:

\[
\frac{dr_k}{dt} = \lambda r_k - r_k^3 + \epsilon \sum_{j=1}^{N} r_j \cos(\theta_j - \theta_k),
\]  

(2.6)

\[
\frac{d\theta_k}{dt} = \omega_k + \epsilon \sum_{j=1}^{N} \frac{r_j}{r_k} \sin(\theta_j - \theta_k).
\]  

(2.7)

Assume that \( \epsilon \ll 1 \) and thus \( \frac{\epsilon}{N} \sum_{j=1}^{N} r_j \cos(\theta_j - \theta_k) = 0 \), (2.6) can be reduced to

\[
\frac{dr_k}{dt} = \lambda r_k - r_k^3
\]  

(2.8)

which is a Bernoulli Equation.

Now we solve (2.8) for all values of \( k \) by letting \( v = r_k^{-2} \),

\[
r_k = v^{-1/2}.
\]  

(2.9)

Differentiate (2.9) with respect to time \( t \) to get

\[
\frac{dr_k}{dt} = (-\frac{1}{2}) v^{-3/2} \frac{dv}{dt}.
\]  

(2.10)

Hence, substitute (2.9) and (2.10) into Eq.(2.8) to obtain

\[
-\frac{1}{2} v^{-3/2} \frac{dv}{dt} - \lambda v^{-1/2} = v^{-3/2}.
\]

Hence dividing it by \(-v^{-3/2}\) so that

\[
\frac{dv}{dt} + 2\lambda v = 2.
\]  

(2.11)
Let \( p(t) = 2\lambda \) and applying \( \mu(t) = e^{\int p\lambda dt} \), we get

\[
\mu(t) = e^{2\lambda t},
\]

\[
v e^{2\lambda t} = \int 2e^{2\lambda t} dt + c,
\]

Assuming \( c = 0 \), solve the integration to get the solution for \( r_k \) as:

\[
v = \frac{1}{\lambda} = \frac{1}{r_k},
\]

\[
r_k \approx \sqrt{\lambda}.
\]

The result above indicate that \( r_k \) is a constant and thus implies the system of oscillators will rotate in a unit cycle. Let \( r_j = r_k \) and \( r_j \approx \sqrt{\lambda} \) to get the Kuramoto Model:

\[
\frac{d\theta_k}{dt} = \omega_k + \frac{\epsilon}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_k).
\]

### 2.2.2 Kuramoto’s Analysis

The simulation made by Kuramoto’s analysis describes an infinite population of coupled oscillators \( \theta_j \) given by the dynamics:

\[
\dot{\theta}_j(t) = \omega_j + \frac{\epsilon}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j), \quad j = 1, \ldots, N,
\]

where \( \epsilon \) as the coupling strength between two oscillators, \( N \) is the number of oscillators, \( \theta_j(t) \) is the phase (angle) of the \( j^{th} \) oscillator at time \( t \), while \( \omega_j \) is its intrinsic frequency randomly drawn from some unimodal distribution \( g(\omega) \) which is symmetric about a mean \( \Omega \), i.e \( g(\Omega + \omega) = g(\Omega - \omega) \). In other definition (Rogge, 2006), a distribution \( g(\omega) \) symmetric about mean \( \Omega \) is unimodal if it is nowhere increasing on \([\Omega, \infty)\), i.e.

\[
\omega_1 \leq \omega_2 \Rightarrow g(\omega_1) \geq g(\omega_2), \quad \forall \omega_1, \omega_2 > \Omega.
\]

Generally a Gaussian distribution is widely used by researchers (Daniels, 2005, Strogatz, 2000), while others use a uniform distribution (Pluchino et al., 2006, 2004) and Cauchy or Lorentzian distribution (Acebron et al., 2005, Jadbabaie et al., 2004). The summation denotes for all the oscillators while the parameter \( \epsilon \geq 0 \) measures the coupling strength. Notice that the factor \( \frac{1}{N} \) is to ensure that
Figure 2.1: Geometric interpretation of the order parameter (2.15).

the model well behaved as $N \to \infty$. The equation depicts a globally coupled system with a mean-field coupling.

Kuramoto defined a complex order parameter to visualize the system through an auxiliary quantity given by:

$$r e^{i\psi} = \frac{1}{N} \sum_{k=1}^{N} e^{i\theta_k}. \quad (2.15)$$

The magnitude $0 \leq r(t) \leq 1$ is a measure of the coherence of the population and $\psi(t)$ is the average phase. It can be interpret geometrically of points rotating around a unit circle in a complex plane (Figure 2.1). We want to rewrite (2.14) in terms of the order parameter. So first we multiply both sides of (2.15) by $e^{-i\theta_j}$ to get

$$r e^{i(\psi - \theta_j)} = \frac{1}{N} \sum_{k=1}^{N} e^{i(\theta_k - \theta_j)}$$

By applying the Euler’s Formula and equating imaginary parts of above equation we obtain

$$r \sin(\psi - \theta_j) = \frac{1}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j).$$

Thus (2.14) yields

$$\dot{\theta}_j(t) = \omega_j + \epsilon r \sin(\psi - \theta_j), \quad i = 1, \ldots, N. \quad (2.16)$$

2.2.3 Steady Solutions by Kuramoto Model

Kuramoto first took an effort to guess the solutions in long-term behaviour when the limit $N \to \infty$. Criteria of this steady solution is where $r$ is assumed constant, implies that all the oscillators are effectively independent, and $\psi(t)$ rotates uni-
formally at frequency $\Omega$. Considering in the rotating frame with $\Omega$ at its origin, it is set $\psi \equiv 0$. With the substitution $\omega_j = \omega_j - \Omega$, the governing equation is now

$$\dot{\theta}_j = \omega_j - \epsilon r \sin \theta_j, \quad j = 1, \ldots, N. \quad (2.17)$$

In the limit of infinitely many oscillators, the system is expected to be distributed with a probability density $\rho(\theta, \omega, t)$ with the normalization condition

$$\int_{-\pi}^{\pi} \rho(\theta, \omega, t) d\theta = 1. \quad (2.18)$$

Expressing (2.15) in terms of the probability density, we may get an average over phase and frequency order parameter

$$r e^{i\psi} = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} e^{i\theta} \rho(\theta, \omega, t) g(\omega) d\omega d\theta. \quad (2.19)$$

This system will obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta} \{[\omega + \epsilon r \sin(\theta - \psi)] \rho\} = 0. \quad (2.20)$$

Solving this continuity equation with (2.18) and (2.19) to obtain

$$\rho(\theta, \omega) = \frac{C}{|\omega - \epsilon r \sin(\psi - \theta)|} \quad (2.21)$$

where

$$C = \frac{1}{2\pi} \sqrt{\omega^2 - (\epsilon r)^2}. \quad (2.22)$$

By this illustration of order parameter, we can measure the oscillator synchronization when the coupling limit $\epsilon$ varies, which resulted in three particular conditions: (i) Not Synchronized, (ii) Partially Synchronized and (iii) Globally Synchronized:

(i) Weak coupling: When $\epsilon \to 0$, Eq. (2.16) gives $\theta_j \approx \omega_j(t) + \theta_j(0), \forall j = 1, \ldots, N$, where subsequently gives $\theta \approx \omega(t)$. Using this to analyse Eq.(2.19), we will find that $r \to 0$ as $t \to \infty$ as we apply the Riemann-Lebesgue lemma. This implies that all oscillators rotate at their natural frequencies and are not synchronized.

(ii) Intermediate coupling: When $\epsilon_c < \epsilon < \infty$, accordingly $0 < r < 1$, where part of the oscillators cluster around the mean phase while the others rotating out
of synchrony with their own frequency. Thus, the oscillators are \textbf{partially synchronized}.

(iii) Strong coupling: When $\epsilon \to \infty$, eventually $\theta_j \approx \psi \ \forall j = 1, \ldots, N$ and Eq. (2.19) denotes $r \to 1$. Thus, all oscillators synchronized to their mean phase, i.e. the oscillators are \textbf{globally synchronized}.

Now we use (2.19) and (2.21) to calculate the order parameter in the partial synchronization state:

\[
\begin{align*}
r &= \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} e^{i(\theta - \psi)} \delta(\theta - \psi - \sin^{-1}(\omega / \epsilon r)) g(\omega) d\omega d\theta \\
&\quad + \int_{-\pi}^{\pi} \int_{|\omega| > \epsilon r} e^{i(\theta - \psi)} \frac{C g(\omega)}{[\omega - r \sin(\theta - \psi)]} d\omega d\theta 
\end{align*}
\]

(2.23)

Assuming that $g(\omega) = g(-\omega)$ (even function), we have the symmetry $\rho(\theta + \pi, -\omega) = \rho(\theta, \omega)$, which implies the second term will equal to zero. Let us express the above equation in the form below:

\[
\begin{align*}
r &= \int_{|\omega| < \epsilon r} \cos(\sin^{-1}(\omega / \epsilon r)) g(\omega) d\omega \\
&\quad + \int_{-\pi/2}^{\pi/2} \cos(\theta) g(\epsilon r \sin \theta) \epsilon r \cos \theta d\theta, \\
&\quad \quad (2.24)
\end{align*}
\]

Corresponding to the incoherence $\rho = (2\pi)^{-1}$, we always will have a trivial solution $r = 0$. However, for partially synchronize phase the solution satisfying the consistency condition on the amplitude of the order parameter:

\[
1 = \epsilon \int_{-\pi/2}^{\pi/2} \cos^2 \theta g(\epsilon r \sin \theta) d\theta
\]

(2.25)

This solution bifurcates continuously from $r = 0$ at $\epsilon = \epsilon_c$ which is the critical coupling:

\[
\epsilon_c = \frac{2}{r g(0)}.
\]

(2.26)

In most general study, a Lorenztian density function is utilize for $g(\omega)$:

\[
g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)}.
\]

(2.27)

This allow the explicit calculations of the integrals with exact result after inte-
grating with (2.25):

\[ r = \sqrt{1 - \frac{\epsilon_c}{\epsilon}} \]

for all \( \epsilon > \epsilon_c = 2\gamma \) where it matches the sketched of Figure 2.3. The solutions of above exhibit two types of long-term behaviour which depends on the size of \( \omega_j \) relative to \( \epsilon r \).

As a summary, there is a critical value \( \epsilon_c \) as a threshold of the coupling strength. This value determined the onset of the population to exhibit synchronization behaviour. The coupling strength \( \epsilon \) is a parameter which we denote as the interaction between the oscillators. The hypothesis based on this parameter is, the stronger the interactions between oscillators, the larger the phase locking group until it reaches fully synchronized state when \( \epsilon \to \infty \) as \( r_\infty \) continue increasing from zero to one (Figure 2.3).
2.3 Opinion Changing Rate Model

The interactions dynamics in most consensus models in sociophysics is governed by very simple deterministic rules. Clearly the goal of opinion dynamics models is to comprehend the timing and state of a consensus to be reached commencing from the initially different opinions. As in the work of Pluchino, Latora and Rapisarda in 2004 (Pluchino et al., 2004), the Kuramoto Model is modified to Opinion Changing Rate Model (OCR) by adding some other terms into the original Kuramoto Model to comply with the study on opinion changing and synchronization. The main interest of the thesis is on the dynamical aspects of the consensus formation instead of the equilibrium one.

The OCR model as in (Pluchino et al., 2004):

\[ \dot{x}_j = \omega_j + \alpha \frac{K}{N} \sum_{k=1}^{N} \sin(x_k - x_j) e^{-\alpha|x_k-x_j|}, \quad j = 1, \ldots, N, \]  

(2.28)

where \( x_j(t) \) is the opinion (denote by certain real number) of the \( j \)th individual at time \( t \). \( \alpha \) is the tuning parameter of the threshold for the exponential factor and also to rescale the range of coupling strength, \( K \) is the interacting strength and \( N \) is the number of individual as in Kuramoto model. Corresponding to the natural frequency of the oscillators in Kuramoto model, here \( \omega_j \) is the natural opinion changing rate, i.e. the intrinsic inclination, or natural tendency of each individual to change his/her opinion. In the study, \( \omega_j \) are taken from a uniform distribution of the rage \([0, 1]\). There are three cases when the values of \( \omega_j \) varies:

(i) \( \omega_j < \omega_0 \): Conservative individuals, who are reluctant to change or only change their opinions gradually.

(ii) \( \omega_j \approx \omega_0 \): Adaptable individuals, may change or may not change their opinions.

(iii) \( \omega_j > \omega_0 \): Flexible individuals, most likely to change their opinions accordingly.

Notice that an exponential term \( e^{-\alpha|x_k-x_j|} \) is added to this model to befit in the desired context. This is used to represent the distance: distance between an individual who persuading other with the receiver. If the distance is higher than a certain threshold, the opinion will no more be able to influence others. In the real situation, the distance may not only represent the topological distance.
but also represents the wane of time.

The opinion here suggest a very general perspective. It may refer to any kind of opinions including political views, investment strategy or even regarding new fashions or new trends. Differ from the original Kuramoto Model, the opinion values taken in this model are non-periodicity. Hence, a different order parameter is adopted relevant to the standard deviation of the opinion changing rate. With $\dot{X}(t)$ as the average over $\dot{x}_k(t)$, the order parameter is:

$$R(t) = 1 - \sqrt{\frac{1}{N} \sum_{k=1}^{N} (\dot{x}_k(t) - \dot{X}(t))^2}$$

(2.29)

whereas

- $R=1$: Synchronized phase
- $R<1$: Incoherent or partially synchronized phase

### 2.4 Friedkin-Johnsen Model

The study of opinion formation process typically carried out in a complex social environment. The following equilibrium model has been the guideline to interpret the social influences in many studies such as the study of social structure and social interaction on individuals by Erbring and Young (Erbring and Young, 1979):

$$Y = WY + XB + U$$

- $Y$: $n \times 1$ vector of outcome scores
- $W$: $n \times n$ matrix of scores on $k$ exogenous variables that may include a constant
- $X$: $n \times k$ matrix of opinion
- $B$: $k \times 1$ vector of coefficients giving the effects of each of the exogenous variables
- $U$: $n \times 1$ vector of residual scores
In 1990, Friedkin and Johnsen (Friedkin and Johnsen, 1990) ground the above model in a social network paradigm which provides elementary simplifying assumptions about the process of opinion change and develop a simple recursive definition for the influence process in 1999 (Friedkin and Johnsen, 1999). Here we briefly describes the theory of Friedkin-Johnsen model for a group of $N$ individuals:

$$x(t+1) = AWx(t) + (I - A)x(1), \quad t = 1, 2, 3, \ldots$$  \hspace{1cm} (2.30)

where $x(t)$ is an $N \times 1$ vector of a person’s initial opinions on a particular matter, $x(t)$ is an $N \times 1$ vector of that person’s opinion at time $t$, $W = [w_{jk}]$ is an $N \times N$ matrix of interpersonal influences where $0 \leq w_{jk} \leq 1$ and $\sum_{k} w_{jk} = 1$, while $A = \text{diag}(a_{11}, a_{22}, \ldots, a_{NN})$ is an $N \times N$ diagonal matrix of the person’s susceptibilities to interpersonal influence on the matter, which $0 \leq a_{jj} \leq 1$.

Applying (2.30) iteratively will obtain

$$x(t) = V(t-1)x(1)$$  \hspace{1cm} (2.31)

where

$$V(t-1) = (AW)^{(t-1)} + \left[ \frac{(t-2)}{(k=0)} (AW)^{(k)} \right] (I - A), \quad t = 2, 3, \ldots,$$  \hspace{1cm} (2.32)

In the long run behaviour, assume that at equilibrium, we have $x(t) = x(\infty)$ exist, then

$$x(\infty) = Vx(1)$$  \hspace{1cm} (2.33)

where

$$V = (I - AW)^{-1}(I - A)$$  \hspace{1cm} (2.34)

More generally, we can obtain (2.33) by (2.31) if $x(\infty)$ exists:

In either case, $V$ is a reduced-form matrix of coefficients to describe the total interpersonal effects that transform the initial opinions into final opinions. The coefficients in $V = [v_{jk}]$ are non-negative, i.e. $0 \leq v_{jk} \leq 1$. In addition, the sums for each row of $V$ is equal to 1 ($\sum_{k} v_{jk} = 1$). Hence, $v_{jk}$ provides the relative weight of the initial opinion of individual $k$ to determine the final opinion of individual $j$ for all $j$ and $k$. If $I - AW$ is non-singular, then $V$ can be derived directly from (2.34). Otherwise, $V$ can be estimated numerically from (2.32) for
a sufficiently large $t$ when $x^{(n)}$ exists.

According to Friedkin and Johnsen, they developed the above theory for the general cases where the interpersonal influences and opinions of the individuals in a group are different. As for the special cases, it includes situations in which those differences are constrained. The special cases may involve the classical situations as following:

1. Status order: An individual is located in a stratified influence network.

2. Conformity: A divergence is coped with a fixed consensus of other individuals, regardless of the acceptance of the divergence.

3. Minority influence: A divergence may change the majority harmonious opinion.

4. Intergroup conflict: The occurrence of disagreement between two party.

2.5 Concluding Remarks

Until recently, works on opinion formation process in the context of Malaysia as a multiracial society country mainly focused on using some social science research methodology such as interview and questionnaire. For this reason, existing study on opinion formation process in the context of Malaysia multiracial society failed to address some issues related to dynamical process. In Malaysia, there were several researches on multiracial society or the opinion formation process in the social science field, but our main focus is on the study on mathematical analysis through synchronization in the opinion formation process in this context, which is in the filed of sociophysics.

Therefore, it is a novel yet crucial effort to investigate whether in the sense of mathematical analysis, there is a rigorous proof for the dynamical process of opinions synchronization in the diversities of Malaysia society. We need to conduct the numerical experiment using appropriate mathematical model but so far we have none. Hence, it is ought to develop a modified version of the models to identify synchronization in the opinion formation process.

Kuramoto model seems to be the suitable model in this study since it provides treatment to infinitely many oscillators, which represent a large capacity
of populations. Meanwhile the different version Kuramoto model, the Opinion Changing Rate model, reviewed in section 2.3, Eq.(2.28) appeared to have the criteria to be applied in our subject of interest. We also carry out examination of the linear model adapting the Friedkin-Johnsen Model to see how the two varies in the dynamics of opinion formation process. However, there is still space for exploration in this model in order to equip our study well.
CHAPTER 3

METHODOLOGY

3.1 Introduction

This chapter starts the main content of our study. We will show in details of the models we utilize and all the related exploration. The key is to examine and analyse the models through the desired context to be operated and present the result of the simulations in the next chapter. The basic steps of our method in this exploration are as follows:

1. Show the validity of the model by proving or examine through previous work.

2. Revise and alter the original model in order to suit our field of interest.

3. Specify the characteristics of the parameters involve.

4. Perform simulations of the models and generate the graph by using Maple software (will be shown in Chapter 4).

5. Discuss and justify the result from numerical experiment.

6. Summarize the expected outcome and conclude.

3.2 Analysis of Kuramoto Model

In this section, we will first explain the method to do simulation for Kuramoto Model, then the establishment of the modified version of Kuramoto Model and show its derivation.
3.2.1 Kuramoto Model

As a first effort in obtaining the suitable and reliable opinion changing model, we first carry out a simulation by using the original Kuramoto Model with the adaptation of our desired properties to visualize the opinion change process in the context of Malaysia multiracial society. Applying the model in our context, we need to redefine the parameters and variables in the Kuramoto Model. The model is now

\[
\dot{x}_j(t) = \omega_j + \frac{K_{jk}}{N} \sum_{k=1}^{N} \sin(x_k - x_j), \quad j = 1, \ldots, N.
\]  

(3.1)

The variable \(\theta\) in Equation 2.14 is replaced with \(x\) to represent opinions instead of angle of oscillator in the original model. Thus \(x_j\) is the opinion of individual \(j\) at time \(t\) represented by real numbers. \(\omega_j\) is the number representing the inclination of change of \(j^{th}\) individual, \(K_{jk}\) is the matrix of interaction strength between each individual with another, replacing \(\epsilon\) in the original model which refer to the coupling strength of oscillators. \(N\) as the number of individuals involve in the system.

The properties and values of the parameters utilize in the model are as following:

- Natural inclination of change \(\omega_j\) is randomly drawn from a Lorentzian distribution of \([0,1]\)
- initial opinion \(x(0)\) is generated randomly from a uniform distribution between 0 and 1 for 100 individuals
- Interacting strength \(K_{jk}\) is represent with an \(N \times N\) matrix generated by random matrix tool in Maple, instead of a real number for the coupling strength in the actual Kuramoto Model. Each element in \(K_{jk}\) is assigned for each term in the dynamical system as a scalar value to assess the strength or the existence of interactions between two individuals with different opinions, i.e. values of \(K_{jk}\) and \(K_{kj}\) are assigned to \(\sin(x_j - x_k)\) and \(\sin(x_k - x_j)\) respectively.
- Population of the system is \(N = 100\).

There are two general cases or situations to take into account in our study
in order to obtain a clearer view of the synchronization process: A community where only intra-community links are present but no inter-community links present and a community with the presence of both inter and intra-community links. In our context of study of opinion change among the multiracial society in Malaysia, thus we will use the term inter-racial and intra-racial interactions. For this model, $K_{jk}$ is generated randomly with the number 0 and 1, where 0 indicates that there is no interactions between the individuals, whilst 1 indicates that the individuals interact with each other. Intuitively, we assumed that 90% of the individuals of the same race will interact with each other and 70% of the individuals from each race will interact with another race.

**Definition 1** The inter-community link is the existence of interactions between two different groups of people in which can be categorized according to race, religion or other criteria of community.

**Definition 2** The intra-community link is the existence of interactions within a specific group of people, either within a same race, same religion or other different categorization of community.

### 3.2.2 Modified Kuramoto Model

Our modification on the Kuramoto Model is due to the context that we focus on, i.e. the opinion changes among multiracial society in Malaysia. Hence the model is modified to the following with all same properties for the parameters as above:

$$\dot{x}_j(t) = \omega_j + \frac{K_{jk} N}{N} \sum_{k=1}^{N} \sin(|x_k - x_j|), \quad j = 1, \ldots, N. \quad (3.2)$$

Notice the modulus notation inserted for the phase difference. The idea of inputting the absolute value notation for the opinion difference in our context, is that the understanding and interactions between two individuals are mutual. In other words, two individuals who communicate with each other will exchange their ideas and opinions. Regardless of how their opinions affecting each other, we believe that the differences between them should be at the same degree from both sides of measure. Thus, the absolute notation ensure that $(x_j - x_k) = (x_k - x_j)$. Another incentive to imply the absolute notation is the logical understanding that our opinions here has no direction whilst the magnitude provide the necessary
value for the system. In next subsection we will prove that the modification made is valid and usable for our study.

3.2.2.1 Derivation of Modified Kuramoto Model

Consider a system of $N$ ordinary differential equations which undergoing Hopf Bifurcation (i.e, state limit cycles) which $N$ also represent the community in Malaysia multiracial society. Start with the normal form coordinates in term of each individual, we have

$$\frac{dz_j}{dt} = (\lambda + i\omega)z_j - |z_j|^2z_j + \frac{K}{N} \sum_{k=1}^{N} z_k,$$  \hspace{1cm} (3.3)

where $\lambda > 0$, $\frac{K}{N}$ is the interaction strength. Convert the normal form as in Eq.(3.3) to polar co-ordinates by substitute $z_j = r_j e^{i\theta_j}$, we will get

$$\left(\frac{dr_j}{dt} + r_j i \frac{dx_j}{dt}\right) e^{i|\theta_j|} = (\lambda + i\omega) r_j e^{i|\theta_j|} - |r_j e^{i|\theta_j|}|^2 r_j e^{i|\theta_j|} + \frac{K}{N} \sum_{k=1}^{N} r_k e^{i|\theta_k|},$$  \hspace{1cm} (3.4)

Divide equation above by $e^{i|\theta_j|}$, we get

$$\frac{dr_j}{dt} + r_j i \frac{dx_j}{dt} = \lambda r_j + i\omega r_j - r_j^3 + \frac{K}{N} \sum_{k=1}^{N} r_k e^{i|\theta_k-x_j|}.$$  \hspace{1cm} (3.5)

Translating the interacting term i.e. $\sum_{k=1}^{N} r_k e^{i|\theta_k-x_j|}$ into triangular form with Euler’s formula as follow,

$$\frac{dr_j}{dt} + r_j i \frac{dx_j}{dt} = \lambda r_j + i\omega r_j - r_j^3 + \frac{K}{N} \sum_{k=1}^{N} \left( r_k \cos(|\theta_k - \theta_j|) + i r_k \sin(|\theta_k - \theta_j|) \right).$$  \hspace{1cm} (3.6)

Again, we can separate the real and imaginary parts from the previous equation, we will get

$$\frac{dr_j}{dt} = \lambda r_j - r_j^3 + \frac{K}{N} \sum_{k=1}^{N} r_k \cos(|\theta_k - \theta_j|),$$  \hspace{1cm} (3.7)

$$\frac{dx_j}{dt} = \omega_j + \frac{K}{N} \sum_{k=1}^{N} \frac{r_k}{r_j} \sin(|\theta_k - \theta_j|),$$  \hspace{1cm} (3.8)
Let $K \ll 1$ and so $\frac{K}{N} \sum_{k=1}^{N} r_k \cos(|x_k - x_j|) = 0$, Eq.(3.7) is reduced to

$$\frac{dr_j}{dt} = \lambda r_j - r_j^3,$$

which is the Bernoulli Equation. Hence we get the solution for $r_j$ as

$$v = \frac{1}{\lambda} = \frac{1}{r_j^2},$$

$$r_j \approx \sqrt{\lambda},$$

By replacing the value of $r_j$ and $r_k$ into the interacting term of the Eq.(3.8), where $r_j \approx \sqrt{\lambda}$ as well as $r_k \approx \sqrt{\lambda}$, we get the modified Kuramoto Equation as follow,

$$\frac{dx_j}{dt} = \omega_j + \frac{K}{N} \sum_{k=1}^{N} \sin(|x_k - x_j|).$$

This is how we derive the modified Kuramoto model.

### 3.3 Opinion Changing Rate Model

The opinion changing rate model is one rare study carried out by researchers which look into the opinion change in a group of people. But, this study could be a very useful reference for us in order to achieve our objectives in this research. The model as stated in section 2.3 is

$$\dot{x}_j = \omega_j + \alpha \frac{K_{jk}}{N} \sum_{k=1}^{N} \sin(x_k - x_j)e^{-\alpha|x_k - x_j|}, \quad j = 1, \ldots, N.$$

$x_j$ is the opinion of $j^{th}$ individual, $\omega_j$ is the individual’s natural inclination of change towards any opinion, $\alpha$ is the tuning constant for the system, $K_{jk}$ represents the interacting strength between 2 individuals and $N$ is the number of individuals present in the system. Here we maintain the $\alpha = 3$ as used in the previous study and for better visualization of the graph, we use time steps 1 with 100 iterations.

To befit our scope of study in this research, the values for certain variables are being altered and some necessary properties are being specified accordingly:
• Natural inclination of change $\omega_j$: drawn randomly from a Cauchy (also known as Lorenzt) distribution of the range $[-1, 1]$.

• Interacting strength $K_{jk}$: a matrix of the dimension $N \times N$ so that each entry, 0 or 1 represents the existence of interaction between individual $j$ and $k$. We also set that the percentage of the individuals who interact with the same race people to be 90% while interactions with different races is 70%.

• Initial opinion $x_0$: generated by uniform distribution of the range $[0, 1]$.

With these properties, we will show the simulation operated on the model in next chapter.

3.3.1 Modified Opinion Changing Rate Model

As the same theory of our modification on Kuramoto Model, we operate the same alteration on OCR to get a more suitable model to fit our scope of study. By inserting the modulus notation in the term that measures the phase difference in OCR, we have:

$$\dot{x}_j = \omega_j + \frac{K_{jk}}{N} \sum_{k=j}^{N} \sin(|x_k - x_j|) e^{-\alpha|x_k - x_j|}, \quad j = 1, \ldots, N.$$  \hspace{1cm} (3.14)

Another reason in modifying the model in this way is that we observed the opinion difference term for the exponential factor has its modulus originally. Hence, the same should applies on the other. We can observe the difference of the original and modified OCR in the next chapter. All the parameters and properties to be used in the simulation remain the same as discussed in previous subsection (section 3.3).

3.4 Runge-Kutta Method

Runge-Kutta techniques of iterative method were developed by German mathematicians C. Runge and M. W. Kutta around 1900. It is one of the important implicit and explicit methods in numerical analysis for finding numerical approximations to the solutions of ordinary differential equations (ODE) since its rate of convergence is $O(h^4)$. Typically, it produces very accurate approximations even
REFERENCES


